Market Liquidity Risk
and
Market Risk Measurement

Yu Tian

Group Risk Analytics

The Royal Bank of Scotland

Faculty of Electrical Engineering, Mathematics and Computer Science

Delft University of Technology

Amsterdam, the Netherlands

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Yu Tian[1]

Supervisors in RBS:
Daily Supervisor:
Dr. Ron Rood
Group Risk Analytics/ Quantitative Review

Head of Quantitative Review:
Dr. Martijn Derix

Supervisor in TU Delft:
Prof. dr. Cornelis W. Oosterlee
Delft Institute of Applied Mathematics

MSc Thesis Committee:
Prof. dr. Cornelis W. Oosterlee
Dr. ir. F.H. van der Meulen
Dr. ir. J.A.M van der Weide
Dr. Ron Rood

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[1]E-mail: y.tian@student.tudelft.nl

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Chapter 1

Introduction

In the financial world, the term risk is usually associated with the possibility of losing money. Three main types of risk can be distinguished (cf. [8]):

1. market risk - the risk of a change in the value of a financial position due to changes in the value of the underlying components on which that position depends, e.g., stock and bond prices, exchange rates, commodity prices, etc. [13];

2. credit risk - the risk that a bank borrower or counterparty will fail to meet its obligations in accordance with agreed terms [15];

3. operational risk - the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events [16].

These three types of risk do not include the full list of risks potentially affecting a financial institution. In the list of financial risks we usually tend to ignore liquidity risk. Most of the time we are not aware of its presence until a financial crisis happens. However, inadequate consideration of liquidity risk may often lead to disastrous consequences.

The main aim of the thesis is to formulate a concept of liquidity risk and to incorporate liquidity risk in market risk measurement. We first review two types of liquidity risk and the relation between liquidity risk and market risk. To achieve our aim, we use a new framework of portfolio theory introduced by Acerbi. A novelty of Acerbi's framework is that portfolio valuation includes a consideration of liquidity risk in portfolio valuation. Under the new framework, the valuation of a portfolio becomes a convex optimization problem. We give some examples of calculation schemes for the convex optimization problem. Equipped with the new portfolio theory, we can quantify market liquidity risk and introduce a new market risk measure which includes the impact of liquidity risk. We end the thesis by giving some possible questions for further study.

Chapter 2

Market Liquidity Risk

2.1 Liquidity and liquidity risk

The recent turmoil in financial markets which began in the middle of 2007 strongly indicates that liquidity is a very important issue for financial institutions to consider. Before the crisis, asset markets like mortgage markets and stock exchange markets were booming, and funding was readily attainable for financial institutions at a low cost. When the economic situation worsened, many types of assets became difficult to sell without a loss. As is shown in Figure 2.1, we infer that before the subprime crisis, liquidity was in good shape and the financial market was booming in 2005 and 2006. When the crisis happened, liquidity conditions became tighter accompanied by a high volatility of asset prices[1]. A similar result can be found in the Asian financial crisis in 1998. All these events emphasize the crucial role of liquidity.[2]

When talking about liquidity, we can distinguish between two kinds of liquidity, i.e., *market liquidity* and *funding liquidity*.

Definition 2.1 (cf. [8, 17])**. Market liquidity** is the ability of a market participant to execute a trade or liquidate a position with little or no cost, risk or inconvenience. **Funding liquidity** is the ability of a bank to fund increases in assets and meet obligations as they come due, without incurring unacceptable losses.

Two kinds of risk are respectively associated with the above liquidity notions: one is *market liquidity risk*, and the other is *funding liquidity risk*.

[1] The meanings of liquidity condition and asset price volatility are explained in the note part in Figure 2.1.

[2] An exception shown in Figure 2.1 is the internet bubble from 20000. The reason might be that the infrastructure of the financial market was not ruined during that period.

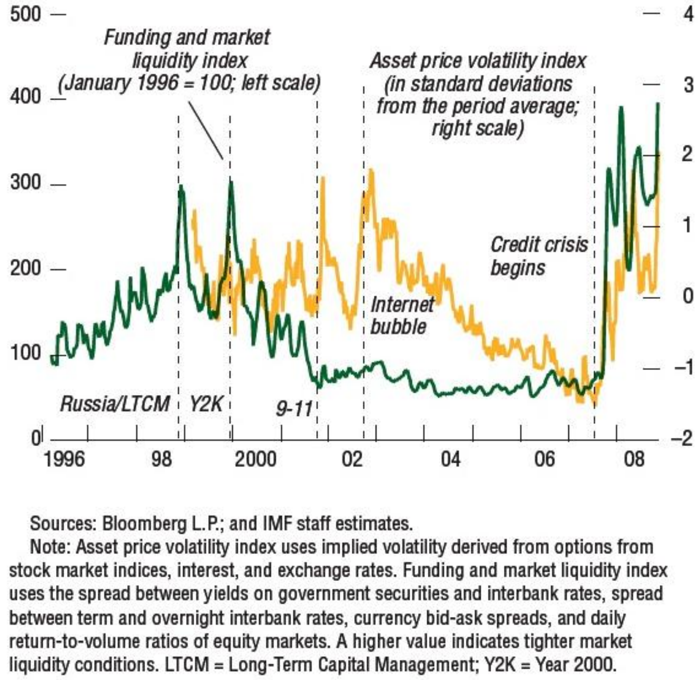


Figure 2.1: Asset price volatility and funding and market liquidity

Definition 2.2. **Market liquidity risk** is the loss incurred when a market participant wants to execute a trade or to liquidate a position immediately while not hitting the best price. **Funding liquidity risk** is the risk that a bank is not able to meet the cash flow and collateral need obligations.

When these two types of liquidity risk occur at the same time, they will give rise to systemic liquidity risk (see [1]), which can be seen as the risk of drainage of liquidity circulating in the whole financial system.

In what follows, we restrict ourselves to a discussion of market liquidity risk.

2.2 Structure of different financial markets

To analyze market liquidity risk, we first look into the trading mechanism which facilitates liquidity to participants. The mechanism is the financial market. A financial market is a mechanism for participants to buy or to sell financial assets. When looking into financial markets, we find that they differ in structure.

One type of the market is an organized market, such as exchanges for stocks. In organized markets, participants typically do not trade directly with each other but quote their buy or sell orders on display for all participants in this market. The availability of all quotes to all market participants increases the chance of participants to find the best prices for their orders. The trading system in exchange market facilitates the determination of the price in real time. We call this mechanism an auction. Through auctions, market participants quote prices to sell or buy an asset. The combination of all sale and purchase orders at a given time can be represented by a list of *ask prices* and *bid prices* with corresponding trading volumes.

Definition 2.3. An **ask price** is the price that a seller is willing to accept for an asset. A **bid price** is the price that a buyer is willing to pay for an asset.

The difference between the lowest ask price and the highest bid price will be called the *bid-ask spread*. Stocks, futures and options are representatives of assets traded in organized markets.

Another type of financial market distinct from an organized market is an over-the-counter (OTC) market. There is usually no auction[3] determining prices for the asset, but participants trade by direct communication. The asset price is determined through bilateral negotiations. Due to lack of information, we often cannot find data to study this type of market easily. Among assets traded in OTC markets are currencies, bonds, swaps, mortgage-backed securities and other derivatives.

The distinction between the above two types of market is not that organized markets are more liquid than OTC markets[4], but that we can get available data from organized markets more easily than from OTC markets. In an exchange market we can quote prices in real time, but in an OTC market we typically cannot.

Loebnitz identifies ways of price discovery for representative asset markets as shown in Table 2.1.

Representative asset	Stocks	Bonds	Currencies	Options	Futures
Price discovery	Auctions	Negotiations	Negotiations	Auctions	Auctions

Table 2.1: Overview of price discovery for different assets

In our research we focus on organized markets such as stock exchanges where price information is relatively easily discovered.

[3]Although for some assets such as bonds there are exchanges, the volume traded in these exchanges is fairly low compared to the volume traded in the OTC market. See [11] for more information.

[4]For example, the foreign exchange (currency) market is an OTC market but is considered to be a quite liquid market.

2.3 Market risk and market liquidity risk

Referring to the definition of market risk in Chapter 1, market risk is usually calculated from is the current market price of a position. However, this market price commonly omits a consideration of liquidity issues, especially when an asset is quite illiquid. When an asset becomes illiquid, it may happen that we cannot find a buyer to buy the asset and consequently, the market risk increases as well. Because of the tendency of market liquidity risk to compound market risk, we should not isolate market liquidity risk from market risk. As a result, we argue that market liquidity risk is an integral part of market risk. Accordingly, market risk measurement should take account of liquidity risk.

Chapter 3

Conventional Market Risk Measurement

In this chapter, we review conventional market risk measures. First we review the definition of risk measure and see how the conventional risk measurement deals with market risk. In section 3.3, we discuss how market liquidity risk methodologies were incorporated in market risk measurement previously.

3.1 Definition of risk measure

Definition 3.1. Let X be a set of random variables. Then a **risk measure** ρ is a function mapping X to the set of all real numbers $\mathbb{R}$, i.e.,

$$\rho : X \to \mathbb{R}$$

Example 3.1 (Standard deviation)**.** If we assume that the return of a portfolio, r, is stochastic, then we define a risk measure as

$$\rho(r) := \sigma_r$$

where σ_r denotes the standard deviation of the return. In Markowitz portfolio theory (cf. [12]), the standard deviation of the return can be simply used for measuring the risk of a portfolio, as we shall see later in Section 4.1.

Example 3.2 (Worst-case replacement value)**.** Let the portfolio Mark-to-Market value V_t to be stochastic at time t. We define a risk measure as

$$\rho(V_t) := \max\{0, 95\text{th percentile of } V_t\}$$

Then $\rho(V_t)$ is referred to as the worst-case replacement value of the underlying portfolio at time t.

Example 3.3 (Value-at-Risk (VaR))**.** Set $P_{s+t}(t) = V_{s+t} - V_s$. We refer to $P_{s+t}(t)$ as the profit and loss (P&L) over horizon t. We then define a risk measure on the set $\{P_{s+t}(t)\}_s$ as

$$\rho(P_{s+t}(t)) := \inf\{x|\mathbb{P}[P_{s+t}(t) < x] \leq 1 - \alpha\}$$

where $\alpha \in (0, 1)$. The risk measure $\rho(P_{s+t}(t))$ is referred to as Value-at-Risk (VaR) [14] and is denoted as $\mathrm{VaR}_\alpha(P_{s+t}(t))$.

Example 3.4 (Expected Shortfall (ES))**.** We define

$$\rho(P_{s+t}(t)) := \mathbb{E}[P_{s+t}(t)|P_{s+t}(t) > \mathrm{VaR}_\alpha(P_{s+t}(t))]$$

where $\alpha \in (0, 1)$. This risk measure is referred to as an Expected Shortfall (ES) [2].

3.2 Coherent risk measure

Up to now we have been talking about risk measures without having reflected on what properties a good risk measure should satisfy. Artzner et al. [4] have proposed a kind of "good" risk measure, which they refer to as a *coherent risk measure*.[1]

Definition 3.2. A **coherent risk measure** is a risk measure, ρ, which satisfies the following four properties (cf. [4]):

1. Monotonicity[2]: for all $x, y \in X$ with $x \geq y$, we have $\rho(x) \leq \rho(y)$.
2. Translational invariance: for all $x \in X$ and $\alpha \in \mathbb{R}$, we have $\rho(x + \alpha) = \rho(x) - \alpha$.
3. Positive homogeneity: for all $\lambda \geq 0$ and all $x \in X$, we have $\rho(\lambda x) = \lambda\rho(x)$.
4. Subadditivity: for all $x, y \in X$, we have $\rho(x + y) \leq \rho(x) + \rho(y)$.

The above four properties can be interpreted as follows:

1. Monotonicity: If one portfolio has higher values than another for every state, its risk measure should be lower.
2. Translational invariance: If we add a certain amount α (say, cash) to our portfolio, the risk measure of our portfolio should decrease by α.

[1] The reason we discuss a coherent risk measure here is that this risk measure will be used to induce a new kind of risk measure, a coherent portfolio risk measure, as we shall see in Section 6.2.2.

[2] Here $x \geq y$ means $x(\omega) \geq y(\omega)$ for all ω in some underlying sample space Ω.

3. Positive homogeneity: If the size of a portfolio does not influence the risk and if the composition of the portfolio does not change, then changing the scale of the portfolio by a factor λ should give rise to the resulting risk measure being multiplied by λ.

4. Subadditivity: The risk measure of a portfolio should not be larger than the sum of risk measures of each component.

We note that VaR is not a coherent risk measure as it is not subadditive.[3] Acerbi [2] has proven that ES is coherent.

3.3 Risk measures with liquidity risk

In this section, we review two existing approaches to quantifying market liquidity risk. For a detailed review of more market liquidity risk models, we recommend [9].

3.3.1 Spread-adjusted approach

Bangia et al. [5] propose a spread-adjusted approach which takes into account liquidity risk, i.e., the liquidity-risk adjusted Value-at-Risk (LAdj-VaR). They argue that liquidity risk can be described by the bid-ask spread and develop a simple liquidity risk add-on to the conventional VaR measure. First, they assume that the bid-ask spread is stochastic and then use the relative spread, S, which is the bid-ask spread divided by the mid-price, for modeling. Moreover, they define the liquidity risk on an average relative spread $\bar{S}$ plus a multiple of the volatility of the relative spread to cover most, say 99%, of the spread distribution (see Figure 3.1).

Accordingly, the LAdj-VaR is defined as

$$\text{LAdj-VaR} := \text{VaR} + \frac{1}{2} Mid \cdot (\bar{S} + a\bar{\sigma})$$

where VaR is the conventional VaR measure derived from the mid-price, Mid denotes the mid-price, $\bar{S}$ is the average of the relative spread S, $\bar{\sigma}$ is the volatility of the relative spread and a is a scaling factor such that we achieve 99% probability coverage of the change in the relative spread.

This model is simple to use[4] as long as we have data on mid-prices and bid-ask spreads, Furthermore, it improves the conventional VaR measure by the inclusion of spreads.

[3] We refer to [8] for counter-examples. However, VaR follows other three properties.

[4] This point might be the main reason why some practitioners still use this method to consider liquidity risk in practice while many more complicated methods have been proposed during the last 10 years.

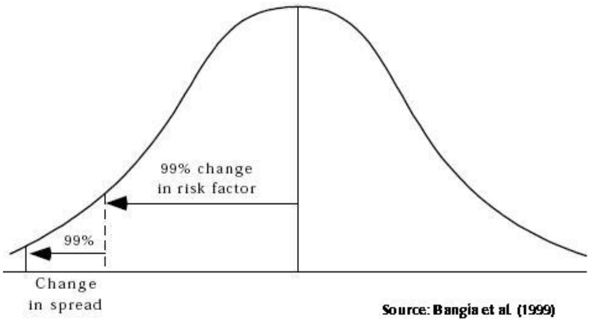


Figure 3.1: Combining market and liquidity risk

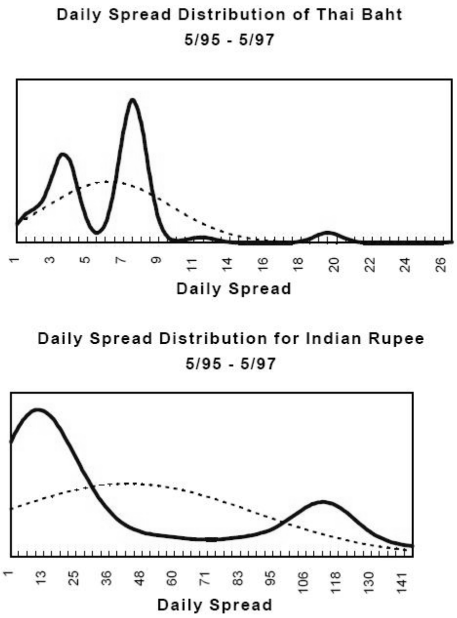


Figure 3.2: Daily spread distribution

However, this model has several weak points. First, the assumption behind this model is that the relative spread should follow some unimodal distribution. In an example from [5], we see that the daily spread distributions for Thai Baht and India Rupee are quite likely to be multimodal distributions (see Figure 3.2). Second, this model only considers the effect of liquidity risk within the bid-ask spread, but it ignores the fact that a large trading size might affect the asset price so as to exceed the spread. As such, we may underestimate the liquidity risk using this model. In addition, a further possible weak point is noticed by Loebnitz [11], who argues that the spread adjustment in LAdj-VaR should be applied to a forecasted mid-price, rather than to an observed mid-price. Thus, he suggests to use a forecasted mid-price to modify the add-on in the LAdj-VaR measure.

3.3.2 Stochastic supply curve approach

Jarrow and Protter [10] propose an adjustment to conventional risk measurement by introducing a stochastic supply curve for security prices as a function of the transaction size. They argue that the position size and direction (buy or sell) of a transaction determine the price of a trade. We mention the major conclusions of their work [10]. The reason why we discuss this approach is mainly because these conclusions somehow give a heuristic background to Acerbi's framework of portfolio theory to be discussed in the next chapter.

The first step in quantifying liquidity risk is to give a functional form of the supply curve. Jarrow and Protter propose a linear supply curve with the slope depending on the state of the economy (crisis or normal). The supply curve, $S(t,x)$, is

$$S(t,x) = S(t,0)[1 + \alpha_c 1_c(t)x + \alpha_n(1 - 1_c(t))x]$$

where $S(t,0)$ is the classical asset price process independent of the trading size (say, geometric Brownian motion), $\alpha_c \geq 0$ and $\alpha_n \geq 0$ are constants and represent crisis and normal situations of the economy, and $1_c(t)$ is an indicator for a crisis[5]. The crisis coefficient α_c should be strictly larger than coefficient α_n, indicating a larger quantity impact on the asset price in times of crisis.

Suppose that, prior to time T, markets are normal, so that we can ignore all liquidation costs before time T. At time T, a crisis happens. Jarrow and Protter assume that an immediate liquidation is required in times of crisis. As such, we have

$$V_T^L = V_T - L_T,$$

where V_T^L is the value of the position when a crisis happens at time T, L_T is the liquidation cost, and V_T is the value of the position without liquidation cost or the impact of trading size. Hence, we have

$$V_T = X_T S(T,0)$$

where X_T is the size of long positions at time T (i.e., $X_T \geq 0$).

[5]That is,

$$1_c(t) = \begin{cases} 1 & \text{if the economy is in crisis at time } t; \\ 0 & \text{if the economy is normal at time } t. \end{cases}$$

Due to a crisis at time T, we have to liquidate $\theta \in [0,1]$ of our position X_T for cash. So the liquidation cost is

$$L_T = -\theta X_T[S(T, -\theta X_T) - S(T, 0)]$$

As there is a crisis at time T, we have

$$S(T, -\theta X_T) = S(T, 0)[1 - \alpha_c \theta X_T]$$

To conclude, we have

$$\begin{aligned} V_T^L &= V_T - L_T \\ &= V_T + \theta X_T[S(T, -\theta X_T) - S(T, 0)] \\ &= X_T S(T, 0) + \theta X_T\{S(T, 0)[1 - \alpha_c \theta X_T] - S(T, 0)\} \\ &= X_T S(T, 0) - \alpha_c \theta^2 X_T^2 S(T, 0) \\ &= V_T[1 - \alpha_c \theta^2 X_T] \leq V_T \end{aligned}$$

For risk management, the key input to a risk measure is the value of the portfolio. We are now equipped with a new formulation of portfolio value incorporating liquidation cost at a time of crisis. We can use risk measures such as VaR, ES and others, based on this formulation which includes liquidity risk. In addition, by the monotonicity of a coherent risk measure, $V_T^L \leq V_T$ implies $\rho(V_T^L) \geq \rho(V_T)$, that is, the risk measure including liquidity risk is greater than that without liquidity risk.

This model is quite appealing in the sense that it includes the impact on asset prices caused by the trading size. A potential deficiency is that the supply curve is assumed to be linear.[6]

[6] See Section 3 in [10] for more information.

Chapter 4

New Framework of Portfolio Theory

4.1 Motivations

Modern portfolio theory, pioneered by H. Markowitz [12] in 1950s, proposed the idea of considering a portfolio from a risk-reward point of view. The reward is described by the expected return of the portfolio and the risk is the standard deviation of the return.

We assume that the return of an asset is stochastic. The return of a portfolio, r, is the proportion-weighted linear combination of the assets' returns r_i ($i = 1, \dots n$), as $r = \sum_{i=1}^{n} w_i r_i$ where $\sum_{i=1}^{n} w_i = 1$. The reward of the portfolio is defined as $\mathbb{E}[r] = \sum_{i=1}^{n} w_i \mathbb{E}[r_i]$. The risk of the portfolio is the standard deviation of the return of the portfolio as $\sigma_r = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_i \sigma_j \rho_{ij}}$ where σ_i is the standard deviation of the return of asset A_i and ρ_{ij} is the correlation between r_i and r_j.

We assume that a rational investor would prefer the portfolio with a lower standard deviation compared to a portfolio with the same expected return but a higher standard deviation. To manage the risk of our portfolio we usually set an upper bound for the standard deviation of the portfolio. We then calculate the weights of our investment in the assets to compose the desired portfolio. See [12] for more information.

However, Markowitz portfolio theory does not specifically define what a portfolio is, and neither does it define what an asset is. Furthermore, the portfolio theory does not take account of liquidity risk. In whole, these deficiencies will make portfolio valuation inappropriate. So the questions in front of us are, how can we define a portfolio and how we can develop a portfolio value model which can accommodate for liquidity risk.

In this chapter we will present a new framework for determining the value of a portfolio proposed by Acerbi [1, 3], which gives a consideration of liquidity risk by introducing a so-called liquidity policy on a portfolio. When following this framework, two fundamental assumptions are listed beforehand:

1. A list of bid and ask prices for an asset can be found at a given time.
2. Any bid price should be lower than any ask price at a given time.

Most of the work in this chapter originates from Acerbi [1, 3].

4.2 Definition of asset

An asset is an object traded in the market (e.g., a security, a derivative or a commodity). We assume that one unit of an asset corresponds to some standardized amount and that an asset is not quoted by a single price, but by the bid and ask prices. In fact, not only one bid price and one ask price are quoted in the market for the same asset at some given time, but a list of many. Each of these prices is associated with a given maximum trading size. The no-arbitrage assumption which we make is that any bid price is lower than any ask price at any given time. An example of a list of bid and ask prices at some given time are shown in Figure 4.1. The figure shows a real-time chart of the order book for the stock ING Group. The left part of the chart gives the lowest 5 ask prices and the highest 5 bid prices with their corresponding maximum trading sizes.

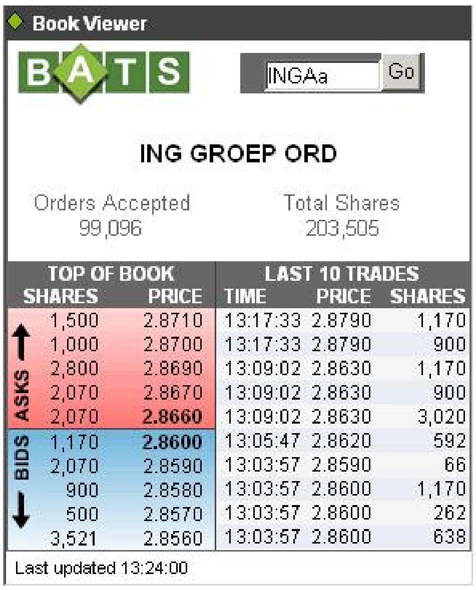


Figure 4.1: A list of bids and asks

To combine the above-mentioned market price information of an asset at a given time, Acerbi introduces a function named *Marginal Supply-Demand Curve (MSDC)* [3]. To build such a function we consider a real-valued variable s which denotes a sale of s units of asset, if $s > 0$, and a purchase of $|s|$ units of asset, if $s < 0$. Here we exclude the case $s = 0$, which means that we will not quote a price for nothing and that no notion of mid-price is considered. The MSDC function $m(s)$ is the last price hit in a trade (sale or purchase) of size s. So, $m(s)$ represents the bid price when $s > 0$ and the ask price when $s < 0$.

To conclude, we have the following definition of *asset*.

Definition 4.1. An **asset** is an object traded in the market characterized by a Marginal Supply-Demand Curve (MSDC) which is defined as a map $m : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying

1. $m(s)$ is non-increasing, i.e., $m(s_1) \geq m(s_2)$ if $s_1 < s_2$;
2. $m(s)$ is càdlàg (i.e., right-continuous with left limits) for $s < 0$ and làdcàg (i.e., left-continuous with right limits) for $s > 0$.

Condition 1 represents the no-arbitrage assumption. Condition 2 ensures that MSDCs, and especially ladder MSDCs as we shall see later, have nice mathematical properties. In contrast with condition 1, we will not heavily use this assumption. Instead, we will mostly make use of the fact that MSDCs are integrable.

As an example, we continue with the above case of ING Group. The MSDC is summarized in Table 4.1. The MSDC can be directly identified from the order book. We can see that the MSDC in this example is non-increasing which follows the above no-arbitrage requirement and is piecewise constant. We refer to such a piecewise constant MSDC as a ladder MSDC. We will consider this more closely in Section 5.2.

	$s \in$	$m(s)$
Asks	[-9440, -7940)	2.8710
	[-7940, -6940)	2.8700
	[-6940, -4140)	2.8690
	[-4140, -2070)	2.8680
	[-2070,0)	**2.8660**
Bids	(0, 1170]	**2.8600**
	(1170, 3240]	2.8590
	(3240, 4140]	2.8580
	(4140,4640]	2.8570
	(4640,8161]	2.8560

Table 4.1: MSDC of ING

We call the limits $m^+ := m(0^+)$ the *best bid* and $m^- := m(0^-)$ the *best ask*. The *bid-ask spread* is denoted by δm and is the difference between the best ask and the best bid, i.e., $\delta m := m^- - m^+$.

We will call a *security* any asset whose MSDC is positive (e.g., a stock, an option, a bond, a commodity) and a *swap* any asset whose MSDC can take both positive and negative values (e.g., a CDS, a repo). Note that any asset with only negative MSDC can be converted to a security-type by taking the absolute value of its MSDC.

A particular example of the asset given by Definition 4.1 is the *cash* asset.

Definition 4.2. [1] **Cash** is the asset representing the currency paid or received when trading any asset. It is characterized by a constant MSDC, $m_0(s) = 1$, (i.e., one unit) for any $s \in \mathbb{R} \setminus \{0\}$.

We say the cash is a *perfectly liquid* asset by the following definition.

Definition 4.3. An asset is called **perfectly liquid** if its MSDC is constant at a given time. Otherwise, it is called **illiquid**.

We can only choose one currency as our cash. For example, if we choose the euro as our cash asset, then the dollar will be seen as an illiquid asset which can be bought or sold at different bid or ask prices. If we choose the dollar as our cash asset, then the opposite is true.

Definition 4.4. The **proceeds** for a transaction of s units of an asset with MSDC m is defined as

$$P(s) := \int_0^s m(x)\mathrm{d}x$$

The *proceeds* is the total money we receive for a sale of s units of one asset and minus the total money we pay for a purchase of s units.

4.3 Portfolio

4.3.1 Definition of portfolio

We define a portfolio as a vector of real numbers. Each term represents the holding volume of the corresponding asset in our portfolio.

Definition 4.5. A **portfolio p** is a vector of real numbers, $\mathbf{p} := (p_0, \overrightarrow{p}) = (p_0, p_1, \ldots, p_N) \in \mathbb{R}^{N+1}$, where $p_i\,(i = 0, 1, \ldots, N)$ are the holding volume of asset A_i. p_0 is the holding volume of cash, which is called the **portfolio liquidity**, and $\overrightarrow{p} = (p_1, \ldots, p_N)$ is the **asset's position**. We call them **long-**, **short-** or **zero-positions** in asset A_k if $p_k > 0$, $p_k < 0$ or $p_k = 0$, respectively.

[1]This definition is modified from the definition of euro in [3] or dollar in [1].

In what follows, we call $\mathcal{P} := \mathbb{R}^{N+1}$ the *portfolio space*. Note that the usual operations of addition and scalar multiplication are valid in $\mathcal{P}$. In Section 4.5 we will see that portfolio values do not have such a linear structure (see Theorem 4.5). We will later write a portfolio $\mathbf{p}$ plus a units of cash asset as $\mathbf{p} + a = (p_0 + a, \overrightarrow{p})$ for simplicity.

Given Definition 4.5, the next question arising is what the value of a portfolio can be. Given N types of illiquid assets, $A_1, \ldots, A_N$, let p_i be the holding volume of asset A_i, $i = 1, \ldots, N$. We thus consider a portfolio $\mathbf{p} = (p_0, p_1, \ldots, p_N)$. First we give some preliminary definitions.

Definition 4.6. The **liquidation Mark-to-Market(MtM) value** of a portfolio $\mathbf{p} \in \mathcal{P}$ is the sum of each proceeds P_i for asset A_i, given by

$$L(\mathbf{p}) := \sum_{i=0}^{N} P_i(p_i) = p_0 + \sum_{i=1}^{N} \int_0^{p_i} m_i(x)\mathrm{d}x$$

$L(\mathbf{p})$ is the total cash we get from the liquidation of all our positions. This situation can be seen as an extreme case where we have to immediately close all positions in our portfolio. The opposite extreme case is to keep our portfolio as it is, i.e., to liquidate nothing.

Definition 4.7. The **uppermost Mark-to-Market(MtM) value** of a portfolio $\mathbf{p} \in \mathcal{P}$ is given by

$$U(\mathbf{p}) := \sum_{i=0}^{N} (m_i^+ \cdot \max(p_i, 0) + m_i^- \cdot \min(p_i, 0)) = p_0 + \sum_{i=1}^{N} (m_i^+ \cdot \max(p_i, 0) + m_i^- \cdot \min(p_i, 0))$$

where m_i^+ and m_i^- are the best bid and the best ask for asset A_i.

Note that $U(\mathbf{p}) \geq L(\mathbf{p})$, as the MSDC is non-increasing. The difference in value between these two extreme cases is called the *uppermost liquidation cost*.

Definition 4.8. The **uppermost liquidation cost** of a portfolio $\mathbf{p} \in \mathcal{P}$ is given by

$$C(\mathbf{p}) := U(\mathbf{p}) - L(\mathbf{p})$$

Remark. $C(\boldsymbol{p}) \geq 0$ *for all portfolio* $\boldsymbol{p} \in \mathcal{P}$ *as* $U(\boldsymbol{p}) \geq L(\boldsymbol{p})$.

4.3.2 Properties of operators U, L and C

In this section we review some useful properties of operators U, L and C that will be used later in Section 4.5.2 and 6.2.2. By definitions of U, L and C (see Section 4.3.1) it is easy to show the following properties.

Proposition 4.1. *Let $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}$ and $\theta \in [0,1]$.*

- *The liquidation MtM value operator $L : \mathcal{P} \to \mathbb{R}$*
  1. *is concave, i.e., $L(\theta\boldsymbol{p} + (1-\theta)\boldsymbol{q}) \geq \theta L(\boldsymbol{p}) + (1-\theta)L(\boldsymbol{q})$.*
  2. *$L(\lambda\boldsymbol{p}) \leq \lambda L(\boldsymbol{p})$ if $\lambda \geq 1$.*
- *The uppermost MtM value operator $U : \mathcal{P} \to \mathbb{R}$*
  1. *is concave, i.e., $U(\theta\boldsymbol{p} + (1-\theta)\boldsymbol{q}) \geq \theta U(\boldsymbol{p}) + (1-\theta)U(\boldsymbol{q})$.*
  2. *is positive homogeneous, i.e., $U(\lambda\boldsymbol{p}) = \lambda U(\boldsymbol{p})$ if $\lambda \geq 0$.*
- *The uppermost liquidation cost operator $C : \mathcal{P} \to \mathbb{R}^+$*
  1. *is convex, i.e., $C(\theta\boldsymbol{p} + (1-\theta)\boldsymbol{q}) \leq \theta C(\boldsymbol{p}) + (1-\theta)C(\boldsymbol{q})$.*
  2. *$C(\lambda\boldsymbol{p}) \geq \lambda C(\boldsymbol{p})$ if $\lambda \geq 1$.*

4.4 Liquidity policy

From the above analysis of the two extreme cases L and U in Section 4.3.1, we infer that the Mark-to-Market value of a portfolio should make sense for different market circumstances. This means that the portfolio value is subject to some liquidity constraints. To give a formal interpretation to these liquidity constraints, the concept of *liquidity policy* is introduced.

Definition 4.9. A **liquidity policy** $\mathcal{L}$ is a closed and convex subset of $\mathcal{P}$ satisfying

1. If $\mathbf{p} = (p_0, \vec{p}) \in \mathcal{L}$ and $a \geq 0$, then $\mathbf{p} + a = (p_0 + a, \vec{p}) \in \mathcal{L}$.
2. If $\mathbf{p} \in \mathcal{L}$, then $(p_0, \vec{0}) \in \mathcal{L}$.

Remark. *As we will see in Section 4.5, following Acerbi's framework, the portfolio valuation becomes a convex optimization problem. From the perspective of optimization, a liquidity policy gives closed and convex constraints to the optimization problem of portfolio value calculation. If the liquidity policy is not closed but the optimal solution lies on the boundary of the domain, then we have to assign $-\infty$ to the portfolio value. To avoid the occurrence of this, we define the liquidity policy to be closed. The convexity ensures that the optimal value we find is unique.*

Example 4.1 (Liquidating-nothing policy)**.** The uppermost MtM value operator U implies such a liquidity policy as the *liquidating-nothing policy*

$$\mathcal{L}^U := \mathcal{P}$$

This liquidity policy means to keep our portfolio as it is.

Example 4.2 (Liquidating-all policy)**.** The liquidation MtM value operator L implies the *liquidating-all policy*

$$\mathcal{L}^L := \{\mathbf{p} = (p_0, \vec{p}) \in \mathcal{P} | \vec{p} = \vec{0}\}$$

This liquidity policy means to liquidate all positions in our portfolio for cash.

Example 4.3 (α-liquidation policy)**.** Let $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in [0, 1]$ for $i = 1, \ldots, N$. A generalized liquidity policy which includes the above two liquidity policies is a so-called *α-liquidation policy*

$$\mathcal{L}^\alpha := \{\mathbf{q} = (q_0, \vec{q}) \in \mathcal{P} | q_0 \geq p_0 + L(\alpha * \vec{p})\}$$

In this definition, $*$ denotes termwise multiplication: $\alpha * \vec{p} = (\alpha_1 p_1, \ldots, \alpha_N p_N)$. $\mathbf{p} = (p_0, \vec{p})$ is any given portfolio in $\mathcal{P}$. This liquidity policy can be seen as a situation when we have to liquidate part of our portfolio for cash at some time.

Example 4.4 (Cash liquidity policy)**.** An example of the liquidity policy which sets a minimum cash requirement for a portfolio is the *cash liquidity policy* as

$$\mathcal{L}(c) := \{\mathbf{p} \in \mathcal{P} | p_0 \geq c \geq 0\}$$

It can be interpreted as the minimum cash requirement so that part of the portfolio can be immediately liquidated to obtain the minimum amount of cash.

We say a liquidity policy $\mathcal{L}_1$ is *more restrictive* than another liquidity policy $\mathcal{L}_2$ if we have $\mathcal{L}_1 \subsetneqq \mathcal{L}_2$. As any liquidity policy $\mathcal{L}$ except $\mathcal{P}$ itself satisfies $\mathcal{L} \subsetneqq \mathcal{P} = \mathcal{L}^U$, so we say the liquidating-nothing policy $\mathcal{L}^U$ is *least restrictive.*

On the contrary, there is no most restrictive liquidity policy. Suppose we have a most restrictive liquidity policy $\mathcal{L}$, then we can define another liquidity policy $\mathcal{L}_a$ as $\mathcal{L}_a =$

$\{\mathbf{p} + a | \mathbf{p} \in \mathcal{P}, a > 0\}$. By the definition of a liquidity policy, we have $\mathcal{L}_a \subsetneq \mathcal{L}$. Hence $\mathcal{L}_a$ is more restrictive than $\mathcal{L}$.

Note that a portfolio is not supposed to satisfy a liquidity policy all the time. The meaning of the policy is that the portfolio will be prepared to satisfy that policy instantaneously if needed, which will be clarified in the next section.

4.5 Portfolio value

4.5.1 Definition of portfolio value

With different liquidity policies, the values of a portfolio should be different. In this section, we will present Acerbi's framework for portfolio value. We first mention the following definition.

Definition 4.10. Given two portfolios $\mathbf{p}, \mathbf{q} \in \mathcal{P}$, a portfolio $\mathbf{q}$ is **attainable** from $\mathbf{p}$, and we write $\mathbf{q} \in Att(\mathbf{p}) \subseteq \mathcal{P}$, if $\mathbf{q} = \mathbf{p} - \mathbf{r} + L(\mathbf{r})$ for some $\mathbf{r} \in \mathcal{P}$.

This means we can obtain portfolio $\mathbf{q}$ from $\mathbf{p}$ by liquidating $\mathbf{r}$ and adding $L(\mathbf{r})$ to the cash.

Now we have the key definition:

Definition 4.11. The **Mark-to-Market (MtM) value** or simply the **value** of a portfolio $\mathbf{p}$ subject to a liquidity policy $\mathcal{L}$ is a function $V^{\mathcal{L}} : \mathcal{P} \to \mathbb{R} \bigcup \{-\infty\}$ defined by

$$V^{\mathcal{L}}(\mathbf{p}) := \sup\{U(\mathbf{q}) | \mathbf{q} \in Att(\mathbf{p}) \bigcap \mathcal{L}\}$$

Let us give an interpretation for this definition:

- If the portfolio $\mathbf{p}$ already satisfies the liquidity policy $\mathcal{L}$ (i.e., $\mathbf{p} \in \mathcal{L}$), then this portfolio can be marked by the uppermost MtM value $U(\mathbf{p})$.
- If $\mathbf{p} \notin \mathcal{L}$ and $Att(\mathbf{p}) \bigcap \mathcal{L} \neq \emptyset$, then we consider all portfolios $\mathbf{q} \in Att(\mathbf{p}) \bigcap \mathcal{L}$ and say that the value of the portfolio $\mathbf{p}$ is the maximum of $U(\mathbf{q})$ for $\mathbf{q} \in Att(\mathbf{p}) \bigcap \mathcal{L}$. And the optimal portfolio $\mathbf{q}^*$ for which $U(\mathbf{q}^*)$ attains its maximum will satisfy $\mathbf{q}^* \in \mathcal{L}$.
- If $\mathbf{p} \notin \mathcal{L}$ and $Att(\mathbf{p}) \bigcap \mathcal{L} = \emptyset$, this means that there is no portfolio attainable from $\mathbf{p}$ that can satisfy $\mathcal{L}$, then we define the portfolio value to be $-\infty$.

Remark. *In the definition we do not want to treat with the case that* $V^{\mathcal{L}}(\boldsymbol{p}) = \infty$ *caused by at least one position* p_i *of our portfolio* $\boldsymbol{p}$ *converging to* ∞*. However, the case that at least one position* p_i *in our portfolio converges to* ∞ *is not realistic and does not make sense in practice. That means that the positions in our portfolio are usually bounded by some natural boundaries, for example, by the limited volumes in the market.*

4.5.2 Properties of portfolio value map $V^{\mathcal{L}}$

The determination of the value of a portfolio corresponds to an optimization problem. However, a critical question arising is how to solve the optimization problem of portfolio value as the constraints in Definition 4.11 are not explicit. First we state the following lemma.

Lemma 4.2. *Let* $\boldsymbol{q} \in Att(\boldsymbol{p})$*. Then* $U(\boldsymbol{q}) \leq U(\boldsymbol{p})$*.*

Then we present the following proposition to transform the constraints into explicit equalities and inequalities.

Proposition 4.3. *Definition 4.11 is equivalent to*

$$V^{\mathcal{L}}(\boldsymbol{p}) = \sup\{U(\boldsymbol{p} - \boldsymbol{r}) + L(\boldsymbol{r}) | \boldsymbol{r} \in \mathcal{P}, \boldsymbol{p} - \boldsymbol{r} + L(\boldsymbol{r}) \in \mathcal{L}\}$$

This optimization problem is convex.

Proof. As in Definition 4.11 the variable $\mathbf{q} \in Att(\mathbf{p})$, we change it to $\mathbf{q} = \mathbf{p} - \mathbf{r} + L(\mathbf{r})$ for some $\mathbf{r} \in \mathcal{P}$. As $\mathbf{q} \in \mathcal{L}$, then we have $\mathbf{p} - \mathbf{r} + L(\mathbf{r}) \in \mathcal{L}$. By definitions of L and U (see Definition 4.6 and 4.7), we have that

$$U(\mathbf{q}) = U(\mathbf{p} - \mathbf{r} + L(\mathbf{r})) = U(\mathbf{p} - \mathbf{r}) + L(\mathbf{r})$$

From the convexity of liquidity policy $\mathcal{L}$ (see Definition 4.9), we know the optimization problem is convex. $\square$

Skipping the case when $V^{\mathcal{L}}(\mathbf{p}) = -\infty$, we can also write this convex optimization problem in Proposition 4.3 as

$$\begin{cases} \max\ U(\mathbf{p} - \mathbf{r}) + L(\mathbf{r}) \\ \text{s.t. } \mathbf{p} - \mathbf{r} + L(\mathbf{r}) \in \mathcal{L} \end{cases}$$

For any portfolio $\mathbf{p}$, no matter which kind of liquidity policy we hold, we have the following proposition.

Proposition 4.4. *If $\mathcal{L}_1 \subsetneqq \mathcal{L}_2$, then $V^{\mathcal{L}_1}(\boldsymbol{p}) \leq V^{\mathcal{L}_2}(\boldsymbol{p})$ for all $\boldsymbol{p} \in \mathcal{P}$. Furthermore, $V^{\mathcal{L}}(\boldsymbol{p}) \leq U(\boldsymbol{p})$ for any liquidity policy $\mathcal{L}$.*

Proof. For liquidity policies $\mathcal{L}_1$ and $\mathcal{L}_2$, we have

$$V^{\mathcal{L}_1}(\mathbf{p}) = \sup\{U(\mathbf{q}_1)|\mathbf{q}_1 \in Att(\mathbf{p}) \bigcap \mathcal{L}_1\}$$

and

$$V^{\mathcal{L}_2}(\mathbf{p}) = \sup\{U(\mathbf{q}_2)|\mathbf{q}_2 \in Att(\mathbf{p}) \bigcap \mathcal{L}_2\}$$

Since $\mathcal{L}_1 \subsetneqq \mathcal{L}_2$, we have $\mathbf{q}_1 \in Att(\mathbf{q}_2)$. By Lemma 4.2, we have $U(\mathbf{q}_1) \leq U(\mathbf{q}_2)$. From the properties of supremum, we obtain $V^{\mathcal{L}_1}(\mathbf{p}) \leq V^{\mathcal{L}_2}(\mathbf{p})$. Since the liquidating-nothing policy $\mathcal{L}^U$ is the least restrictive, we have $V^{\mathcal{L}}(\mathbf{p}) \leq U(\mathbf{p})$ for any liquidity policy $\mathcal{L}$. $\square$

An important characterization of the portfolio value map $V^{\mathcal{L}}$ is provided by the following theorem.

Theorem 4.5. *Let $\mathcal{L}$ be any liquidity policy. Then we have the following properties of the map $V^{\mathcal{L}}$:*

1. *It is concave, i.e., for any $\theta \in [0, 1]$ and for any $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}$,*

$$V^{\mathcal{L}}(\theta\boldsymbol{p} + (1-\theta)\boldsymbol{q}) \geq \theta V^{\mathcal{L}}(\boldsymbol{p}) + (1-\theta)V^{\mathcal{L}}(\boldsymbol{q})$$

2. *It is translationally supervariant, i.e., for any $\boldsymbol{p} \in \mathcal{P}$ and for any $a \geq 0$,*

$$V^{\mathcal{L}}(\boldsymbol{p} + a) \geq V^{\mathcal{L}}(\boldsymbol{p}) + a$$

Proof. Let $\mathbf{r}$ and $\mathbf{s}$ be the solutions to the optimization problem for $V^{\mathcal{L}}(\mathbf{p})$ and $V^{\mathcal{L}}(\mathbf{q})$, respectively. Hence, we have $V^{\mathcal{L}}(\mathbf{p}) = U(\mathbf{p}-\mathbf{r}) + L(\mathbf{r})$ and $V^{\mathcal{L}}(\mathbf{q}) = U(\mathbf{q}-\mathbf{s}) + L(\mathbf{s})$. From the concavity of U and L (see Proposition 4.1), we have

$$U(\theta(\mathbf{p}-\mathbf{r}) + (1-\theta)(\mathbf{q}-\mathbf{s})) \geq \theta U(\mathbf{p}-\mathbf{r}) + (1-\theta)U(\mathbf{q}-\mathbf{s})$$

and

$$L(\theta\mathbf{r} + (1-\theta)\mathbf{s}) \geq \theta L(\mathbf{r}) + (1-\theta)L(\mathbf{s})$$

Combining the above two inequalities and taking the supremum, we get the first result.

We note that $U(\mathbf{p}+a-\mathbf{r}) + L(\mathbf{r}) = U(\mathbf{p}-\mathbf{r}) + L(\mathbf{r}) + a$. By the definition of liquidity policy (see Definition 4.9), we infer that the feasible set of the optimization for $V^{\mathcal{L}}(\mathbf{p})$ is the subset of that of the optimization for $V^{\mathcal{L}}(\mathbf{p}+a)$. Hence the optimal value of $V^{\mathcal{L}}(\mathbf{p}+a)$ is not less than that of $V^{\mathcal{L}}(\mathbf{p}) + a$. $\square$

Theorem 4.5 shows an important principle that blending two portfolios into one generates additional value.

Chapter 5

Examples of MSDC Models

In previous chapter we have reviewed Acerbi's framework for portfolio valuation, and we find that portfolio valuation corresponds to a convex optimization problem. We now address the following questions:

1. How can we model the MSDC in practice?
2. How can we calculate portfolio value effectively?

In this chapter, we will provide answers to these questions.

5.1 Continuous MSDCs

We set our liquidity policy to be the cash liquidity policy $\mathcal{L}(c)$. The value of a given portfolio can be determined by solving the following convex optimization problem:

$$\begin{cases} \max\ U(\mathbf{p}-\mathbf{r})+L(\mathbf{r}) \\ \text{ s.t. } p_0-r_0+L(\mathbf{r})=c \end{cases}$$

Note that the cash component r_0 of $\mathbf{r}$ plays no role in the optimization problem, as the equations

$$U(\mathbf{p}-\mathbf{r})+L(\mathbf{r})=p_0+\sum_{i=1}^{N}(m_i^+\max(p_i-r_i,0)+m_i^-\min(p_i-r_i,0))+\sum_{i=1}^{N}\int_0^{r_i}m_i(x)\mathrm{d}x$$

and

$$\mathbf{p}-\mathbf{r}+L(\mathbf{r})=(p_0,\vec{p}-\vec{r})+\sum_{i=1}^{N}\int_0^{r_i}m_i(x)\mathrm{d}x$$

do not depend on r_0. So we may as well take $r_0 = 0$ for simplicity.

The above convex optimization problem can hence be written as

$$\begin{cases} \max\ U(\mathbf{p}-\mathbf{r}) + L(\mathbf{r}) \\ \text{s.t. } L(\mathbf{r}) = c - p_0 \end{cases}$$

In Section 5.1.1 we study this optimization problem by making assumptions on the underlying MSDC. Note that the cash liquidity policy yields an equality constraint only. We will study general equality and inequality constraints later in Section 5.1.3.

5.1.1 Continuous MSDCs in general

We first assume that the MSDCs $m_i(s)$ are continuous on $\mathbb{R}$ (i.e., $m_i(0)$ are supposed to exist) and are strictly decreasing for all $i = 1, \ldots, N$.

To obtain the solution to the optimization problem for portfolio valuation characterized by such MSDCs, we have the following proposition.

Proposition 5.1. *The solution $\boldsymbol{r}^{opt} = (0, \vec{r}^{opt})$ to the above convex optimization problem with the continuous MSDCs and the cash liquidity policy is unique and is given by*

$$r_i^{opt} = m_i^{-1}(\frac{m_i(0)}{1+\lambda}), \quad \text{if } p_0 < c$$
$$r_i^{opt} = 0, \quad \text{if } p_0 \geq c$$

where m_i^{-1} denote the inverse of the MSDC function m_i and λ is the Lagrange multiplier.

Proof. We use the Lagrange multiplier method. The case $p_0 \geq c$ is trivial as $\mathbf{p} \in \mathcal{L}(c)$ and hence $V^{\mathcal{L}(c)}(\mathbf{p}) = U(\mathbf{p})$.

Consider the case $p_0 < c$. The original convex optimization problem can be written as

$$\begin{cases} \min\ -U(\mathbf{p}-\mathbf{r}) - L(\mathbf{r}) \\ \text{s.t. } -L(\mathbf{r}) + c - p_0 = 0 \end{cases}$$

The function $-L(\mathbf{r}) + c - p_0$ is convex with respect to variable $\mathbf{r}$. We introduce an auxiliary function

$$G(\mathbf{r}, \lambda) = -U(\mathbf{p}-\mathbf{r}) - L(\mathbf{r}) - \lambda[L(\mathbf{r}) - c + p_0]$$

and solve the equations $\frac{\partial G(\mathbf{r},\lambda)}{\partial r_i} = 0$. For any $i \in \{1, \ldots, N\}$, we have

$$
\begin{aligned}
&\frac{\partial}{\partial r_i}[-U(\mathbf{p}-\mathbf{r}) - L(\mathbf{r}) - \lambda(L(\mathbf{r}) - c + p_0)] = 0 \\
\Longleftrightarrow&\frac{\partial}{\partial r_i}[-p_0 - \sum_{i=1}^{N} m_i(0)(p_i - r_i) - \sum_{i=1}^{N}\int_0^{r_i} m_i(x)\mathrm{d}x - \lambda(\sum_{i=1}^{N}\int_0^{r_i} m_i(x)\mathrm{d}x - c + p_0)] = 0 \\
\Longleftrightarrow& m_i(0) - (1+\lambda)m_i(r_i) = 0 \\
\Longleftrightarrow& r_i^{opt} = m_i^{-1}(\frac{m_i(0)}{1+\lambda})
\end{aligned}
$$

The Lagrange multiplier λ can be found from the equation $L(\mathbf{r}) = c - p_0$. □

Remark. *Since we have*

$$\frac{\partial L(\boldsymbol{r})}{\partial r_i} = m_i(r_i)$$

and

$$\frac{\partial C(\boldsymbol{r})}{\partial r_i} = m_i(0) - m_i(r_i)$$

and then from the equation $m_i(0) - (1+\lambda)m_i(r_i) = 0$, *we find that*

$$
\begin{aligned}
&\frac{\partial C(\boldsymbol{r})}{\partial r_i} - \lambda\frac{\partial L(\boldsymbol{r})}{\partial r_i} = 0 \\
\Longleftrightarrow&\lambda = \frac{\mathrm{d}C}{\mathrm{d}L}
\end{aligned}
$$

We see that the Lagrange multiplier λ can be interpreted as the marginal cost of the liquidation.

For practical modeling purposes, we can extend the continuous MSDC to the case where the MSDC is not continuous at the point 0, i.e., to the case where the bid-ask spread exists. We only need to change the point $m_i(0)$ to the limit m_i^+.

5.1.2 Exponential MSDCs

In this section, we continue our discussion of continuous MSDCs by looking at exponential MSDCs. It turns out that exponential MSDCs form an effective means of characterizing the asset and determining portfolio value by way of convex optimization. We will discuss this in Section 5.3.1.

We consider that there are N illiquid assets characterized by exponential MSDCs $m_i(s) = M_i e^{-k_i s}$ with $M_i, k_i > 0$ for all $i = 1, \ldots, N$. Using this we have

$$U(\mathbf{p}) = p_0 + \sum_{i=1}^{N} m_i(0) p_i = p_0 + \sum_{i=1}^{N} M_i p_i$$

and

$$L(\mathbf{p}) = p_0 + \sum_{i=1}^{N} \int_0^{p_i} m_i(x) \mathrm{d}x = p_0 + \sum_{i=1}^{N} \frac{M_i}{k_i}(1 - e^{-k_i p_i})$$

We compute the Mark-to-Market value of a portfolio with long positions $p_i > 0$ for all $i = 1, \ldots, N$ under the cash liquidity policy $\mathcal{L}(c)$ and assume that $p_0 < c$. Following Proposition 5.1, we have

$$r_i^{opt} = \frac{\log(1+\lambda)}{k_i}, \quad \text{for } \; i = 1, \ldots, N,$$

with

$$\lambda = \frac{c - p_0}{\sum_{i=1}^{N} \frac{M_i}{k_i} - c + p_0}$$

Hence, the portfolio value under the cash liquidity policy equals

$$V^{\mathcal{L}(c)}(\mathbf{p}) = U(\mathbf{p} - \mathbf{r}^{opt}) + L(\mathbf{r}^{opt}) = \sum_{i=1}^{N} M_i (p_i - \frac{\log(1+\lambda)}{k_i}) + c$$

Example 5.1 (Impact of the liquidity risk factor)**.** In exponential MSDCs, $\vec{M} = (M_1, \ldots, M_N)$ can be seen as the market risk factor and $\vec{k} = (k_1, \ldots, k_N)$ as the liquidity risk factor. Thus, we can construct different portfolio value models to see how liquidity risk influences the value of a portfolio, and therefore, we make the following statements:

1. When $\vec{M}$ is multivariate normally distributed and $\vec{k} = \vec{0}$, then the portfolio value model is a Gaussian model with perfect liquidity.

2. When $\vec{M}$ is multivariate normally distributed with constant $\vec{k} \neq \vec{0}$, then the portfolio value model is a Gaussian model with constant liquidity risk factor. However, the portfolio value distribution shifts to the left compared to the distribution without liquidity risk.

3. When $(\vec{M}, \vec{k})$ is multivariate normally distributed and $\vec{k}$ is independent of $\vec{M}$, then the shape of the portfolio value distribution is significantly different from the normal distribution.

4. When $(\vec{M}, \vec{k})$ is multivariate normally distributed and $\vec{k}$ is negatively correlated[1] with $\vec{M}$, then the portfolio value distribution becomes more dispersed.

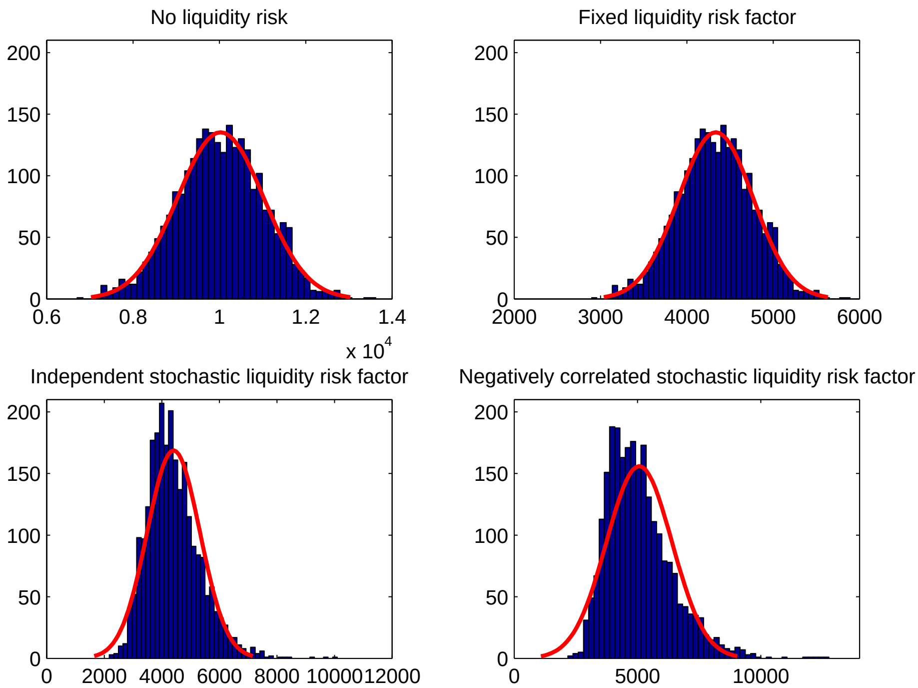
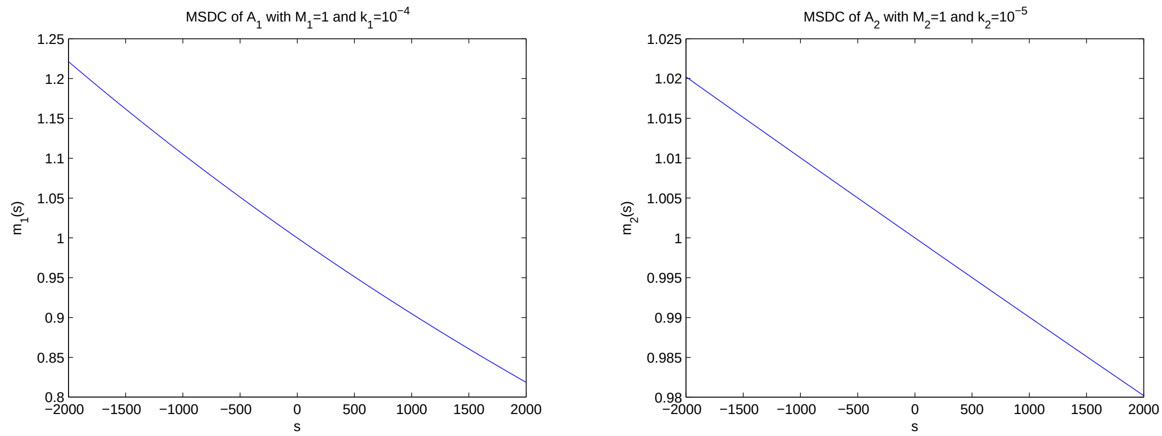


Figure 5.1: Different portfolio value distributions

The above results are illustrated in Figure 5.1 generated by Monte Carlo simulation. The red lines are normal distributions with the parameters estimated from the corresponding portfolio value distributions.

Example 5.2 (A portfolio of two illiquid assets with continuous MSDCs)**.** We continue with exponential MSDCs, and find that the larger the liquidity risk factor k_i is, the more illiquid the corresponding asset is. Suppose, for example, that there are two illiquid assets A_1, with parameters $M_1 = 1$ and $k_1 = 10^{-4}$, and A_2, with $M_2 = 1$ and $k_2 = 10^{-5}$. The MSDCs for these two assets are depicted in Figure 5.2. We can see that the the MSDC of asset A_1 has a wider range than the MSDC of asset A_2 for a same range of trading units.

[1]The idea is that: when the asset price is expected to go up, many investors are willing to buy such an asset, and thus this will increase the liquidity of the asset; when the price is expected to go down investors are reluctant to buy the asset and thereby increase the liquidity risk. However, this result need to be confirmed by more empirical study in the future.



(a) MSDC of A_1 with $M_1 = 1$ and $k_1 = 10^{-4}$

(b) MSDC of A_2 with $M_2 = 1$ and $k_2 = 10^{-5}$

Figure 5.2: MSDCs of A_1 and A_2

We then use this two-asset model to verify the properties in Proposition 4.1. The concavity of the operators U and L and the convexity of operator C are illustrated in Figure 5.3.

Suppose that our portfolio is $\mathbf{p} = (0, 1000, 1000)$. The uppermost MtM portfolio value and the liquidation MtM portfolio value are equal to:

$$U(\mathbf{p}) = p_0 + \sum_{i=1}^{2} M_i p_i = 2000,$$

and

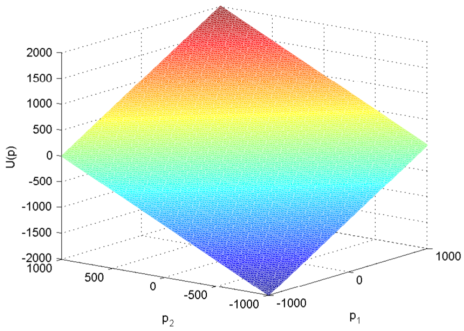
$$L(\mathbf{p}) = p_0 + \sum_{i=1}^{2} \frac{M_i}{k_i}(1 - e^{-k_i p_i}) = 1946.6$$

Suppose that the liquidity policy is the cash liquidity policy, $\mathcal{L}(c) = \{\mathbf{p} | p_0 \geq c, c = 1000\}$. Then we obtain the solution

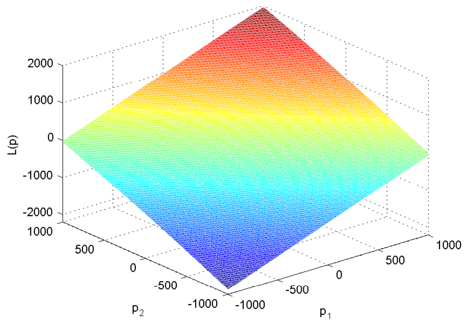
$$\lambda = \frac{c - p_0}{\sum_{i=1}^{2} \frac{M_i}{k_i} - c + p_0} = 9.2 \times 10^{-3},$$

$$r_1^{opt} = \frac{\log(1 + \lambda)}{k_1} = 913.25,$$

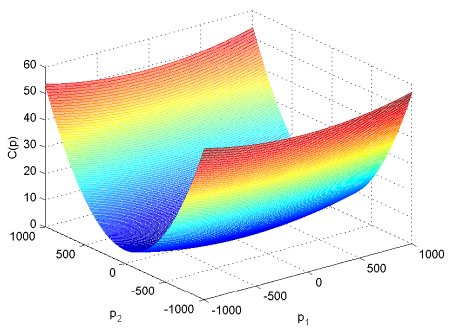
$$r_2^{opt} = \frac{\log(1 + \lambda)}{k_2} = 91.32,$$



(a) Concavity of operator U



(b) Concavity of operator L



(c) Convexity of operator C

Figure 5.3: Different properties of operators U, L and C

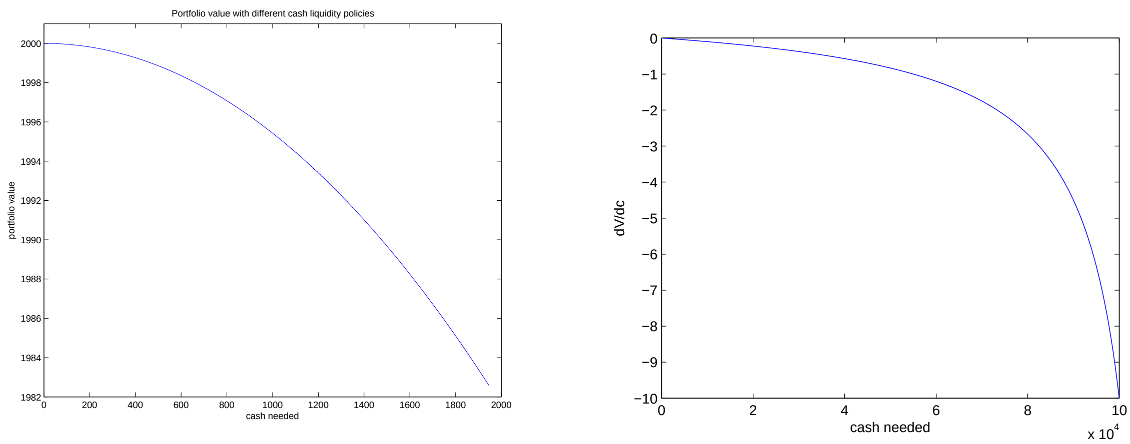
and the corresponding portfolio value reads

$$V^{\mathcal{L}(c)}(\mathbf{p}) = \sum_{i=1}^{2} M_i(p_i - \frac{\log(1+\lambda)}{k_i}) + c = 1995.4$$

From Figure 5.4(a), we see that the portfolio value will decrease when we need more cash. From the partial derivative of V with respect to c,

$$\frac{\partial V}{\partial c} = \frac{-c}{\sum_{i=1}^{2} \frac{M_i}{k_i} - c + p_0} = \frac{-c}{1.1 \times 10^5 - c},$$

we can moreover infer that when the cash needed increases the portfolio value is decreasing at a faster rate. See Figure 5.4(b).



(a) Portfolio value versus cash needed

(b) Decreasing speed $\frac{\partial V}{\partial c}$ vs cash needed

Figure 5.4: Portfolio values under the cash liquidity policy

In financial-economic terms, this can be understood as follows. When we have a liquidity policy which urges us to gather more cash, we have to be prepared to liquidate more positions. As the assets in our portfolio are illiquid, the liquidation makes the sale of assets hitting lower bid prices and thus makes our portfolio value decreasing compared to the uppermost MtM value. Moreover, when we need to liquidate more positions, we first liquidate the most liquid part of our portfolio for cash and this will cause a loss. When we need more cash, the liquidation of more illiquid part of the portfolio makes the loss being much larger and thereby makes the portfolio value decreasing faster.

Example 5.3 (Properties of the portfolio value map)**.** The concavity of the portfolio value map is illustrated in the special cases of U and L in Figure 5.3(a) and Figure 5.3(b). To check the result of translational supervariance in Theorem 4.5, we experiment with an exponential MSDC. We add an amount of cash to portfolio $\mathbf{p}$. The new portfolio value $V(\mathbf{p}+a)$ under the cash liquidity policy reads

$$V^{\mathcal{L}(c)}(\mathbf{p}+a) = \sum_{i=1}^{N} M_i(p_i - \frac{\log(1+\lambda)}{k_i}) + c$$

with

$$r_i^{opt} = \frac{\log(1+\lambda)}{k_i}, \text{ for } i = 1, \ldots, N$$

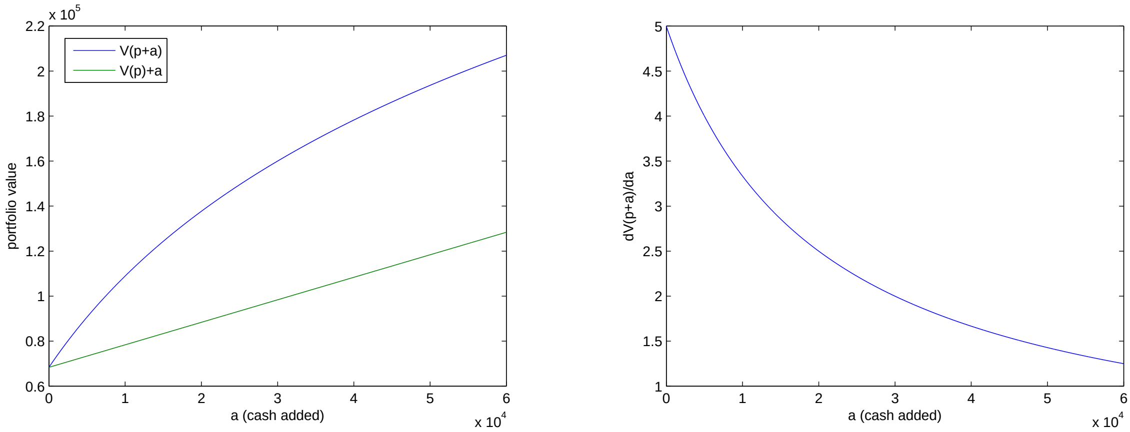
and

$$\lambda = \frac{c - p_0 - 2a}{\sum_{i=1}^{N} \frac{M_i}{k_i} - c + p_0 + 2a}$$

We assume that there is only one illiquid asset in our portfolio with parameters $M_1 = 1$ and $k_1 = 10^{-5}$. The new portfolio is $\mathbf{p} + a = (a, 100000)$ and the cash liquidity policy is $\mathcal{L}(c)$, $c = 60000$ (here we suppose $a \leq c$). Then $V(\mathbf{p} + a)$, and the value $V(\mathbf{p}) + a$ are shown in Figure 5.5(a). We thus confirm the translationally supervariant property of the portfolio value map (see Theorem 4.5), from this example. The derivative of $V(\mathbf{p} + a)$ with respect to a,

$$\frac{\partial V(\mathbf{p} + a)}{\partial a} = \frac{2\sum_{i=1}^{N} \frac{M_i}{k_i}}{\sum_{i=1}^{N} \frac{M_i}{k_i} - c + p_0 + 2a}$$

is depicted in Figure 5.5(b). From Figure 5.5(a), we conclude that a single unit of cash added in our portfolio can generate more value than one unit to our portfolio. We also find that the first unit of cash added to the portfolio is the most valuable and the marginal value of the added cash is decreasing afterwards (see Figure 5.5(b)).



(a) Portfolio value vs cash added

(b) $\frac{\partial V(\mathbf{p}+a)}{\partial a}$ versus added cash

Figure 5.5: Translational supervariance of the portfolio value map

Example 5.4 (Non-linearity)**.** If we assume that there is only one illiquid asset in our portfolio characterized by an exponential MSDC with parameters $M_1 = 1$ and $k_1 = 10^{-4}$, and our liquidity policy is the liquidating-all policy $\mathcal{L}^L$, then the value of the portfolio $\mathbf{p} = (0, p_1)$ versus the size of p_1 is depicted in Figure 5.6. We can find a nonlinear relation between the portfolio size and the portfolio value.

Merits and shortcomings of exponential MSDCs. The use of the exponential MSDCs for modeling is easy for calculation and it gives the price a lower bound of 0, and hence is applicable to securities. For relatively liquid stock exchanges such as the

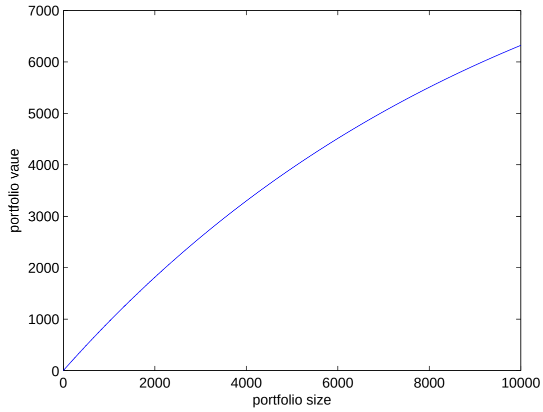


Figure 5.6: Portfolio value vs portfolio size under the liquidating-all policy

New York Stock Exchange and the London Stock Exchange, the liquidity risk factor in the exponential MSDC for the listed stock is usually estimated at the level of 10^{-8} to 10^{-7}. The result is confirmed by an experiment with real market data and by the method of least squares (see Section 5.3.1 for information). The exponential MSDC is a basic model, but may have some deficiencies. For example, the model may work only for bid prices (i.e., long positions), since it results in a steep slope for the part of ask prices without an upper bound, while the lower bound for bid prices is 0.

5.1.3 General liquidity policy

We have seen that the cash liquidity policy gives rise to an equality constraint only (cf. Section 5.1.1). Using the Lagrange multiplier method, one can also generalize the method to any possible liquidity policy which only gives equality constraints. When using a liquidity policy which gives both equality and inequality constraints, one can generalize the method for this kind of optimization problem to a Lagrange duality problem.

To be specific, we transform the inexplicit convex constraints corresponding to a liquidity policy to some inequalities and equalities: $f_j(\mathbf{r}) \leq 0$ $(j = 1, \ldots, m)$ and $h_k(\mathbf{r}) = 0$ $(k = 1, \ldots, p)$ where $f_j(\mathbf{r})$ and $h_k(\mathbf{r})$ are convex functions. Then the convex optimization problem is

$$\begin{cases} \min \ -U(\mathbf{p} - \mathbf{r}) - L(\mathbf{r}) \\ \text{s.t. } f_j(\mathbf{r}) \leq 0, \ j \in \{1, \ldots, m\} \\ \qquad h_k(\mathbf{r}) = 0, \ k \in \{1, \ldots, p\} \end{cases}$$

We then introduce an auxiliary function as

$$G(\mathbf{r}, \overrightarrow{\lambda}, \overrightarrow{\nu}) = -U(\mathbf{p}-\mathbf{r}) - L(\mathbf{r}) + \sum_{j=1}^{m} \lambda_j f_j(\mathbf{r}) + \sum_{k=1}^{p} \nu_k h_k(\mathbf{r})$$

We define the Lagrange dual function as

$$g(\overrightarrow{\lambda}, \overrightarrow{\nu}) = \inf_{\mathbf{r}\in\mathcal{P}} G(\mathbf{r}, \overrightarrow{\lambda}, \overrightarrow{\nu})$$

Then the Lagrange duality problem is

$$\begin{cases} \max \ g(\overrightarrow{\lambda}, \overrightarrow{\nu}) \\ \text{s.t. } \lambda_j \geq 0, \ j \in \{1, \dots, m\} \end{cases}$$

Sometimes solving the duality problem may be easier than solving the primary problem. This may improve the efficiency of portfolio valuation. See [6] for more details on Lagrange duality.

5.2 Ladder MSDCs

In a real market, the MSDC function is typically piecewise constant. We name such a MSDC a ladder MSDC. Ladder MSDCs have limited depth in the sense that quoted sizes are finite at a given time . This implies that we cannot liquidate more positions than the total number of asset units traded in the market in order to calculate the portfolio value. As such, we need to add additional constraints to the convex optimization problem for portfolio valuation.

First suppose that we have a portfolio $\mathbf{p} = (p_0, p_1, \dots, p_N)$. p_i^{max} $(i = 1, \dots, N)$ denotes the maximum trading size for asset A_i in the market at a given time. Our portfolio $\mathbf{p}$ should satisfy $p_i \leq p_i^{max}$ for $i = 1, \dots, N$. Hence, the optimization for portfolio valuation with ladder MSDCs under the cash liquidity policy $\mathcal{L}(c)$ is written as

$$\begin{cases} \max \ U(\mathbf{p}-\mathbf{r}) + L(\mathbf{r}) \\ \text{s.t. } L(\mathbf{r}) = c - p_0 \\ \quad\quad 0 \leq r_i \leq p_i^{max}, \ i = 1, \dots, N \end{cases}$$

We now discuss a way of solving this optimization problem. In particular, we continue to use the cash liquidity policy. If we have a portfolio with long positions $p_i > 0$ for all $i = 1, \dots, N$ in assets whose MSDC are ladder MSDCs, we can directly solve the convex optimization problem numerically, for example, by the *fmincon* function in Matlab. However, this *fmincon* function may give us an approximation of the optimal solution but could be computationally inefficient.

A calculation scheme for portfolio valuation with ladder MSDCs. We here propose a fast and accurate calculation scheme to compute the portfolio value under the cash liquidity policy. The idea is to liquidate the most liquid parts of the portfolio sequentially until the needed cash is obtained.

Suppose that there are N assets A_i ($i = 1, \ldots, N$), each of which is governed by a ladder MSDC with a finite number, say K_i ($i = 1, \ldots, N$), of maximum bid sizes $\Delta s_i^k > 0, k = 1, \ldots, K_i$ (and we have $\sum_{k=1}^{K_i} \Delta s_i^k = s_i$ where $s_i > 0$ denotes the number of contracts traded for asset A_i).

First, we introduce a sensitivity-style function named *marginal sensitivity*, MS_i, with respect to s_i units of asset A_i:

$$MS_i(s_i) := \frac{m_i(0^+) - m_i(s_i)}{m_i(0^+)}$$

The marginal sensitivity is the relative difference between the best bid and the MSDC corresponding to the number of asset units liquidated, and it thus measures the liquidity of asset A_i at s_i units traded. Furthermore, as the MSDC is non-increasing, the marginal sensitivity is non-decreasing for each asset. As for real data the MSDCs are piecewise constant, each marginal sensitivity corresponds to a maximum size Δs_i^k. For a security-type asset, the marginal sensitivity lies in [0,1] as the lower bound for the bid part of the MSDC is 0.

We now introduce a concept which is the *liquidation sequence*. We sort marginal sensitivities in an ascending order with their maximum sizes. As the parts of MSDCs corresponded with the best bids have a zero marginal sensitivity, we first start liquidation from this part of each MSDC to fulfill the cash liquidity policy. This does not change the portfolio value, which means that an infinite number of optimal solutions to the convex optimization problem exist. If we need additional cash, we can later liquidate the second most liquid part of MSDC after liquidating those parts with zero marginal sensitivity and so forth until the cash requirement is satisfied.

As such, a liquidation sequence shows an effective direction for searching the optimal solution, we can derive the optimal solution and calculate the portfolio value more effectively than directly solving the convex optimization problem.

Remark. *Although the example we use here concerns a market with limited depth, this scheme can be extended to the case when the market is unlimited and where the assets are still characterized by ladder MSDCs.*

Example 5.5 (A portfolio with four illiquid assets)**.** Suppose that there are four illiquid assets. The bid prices with limited maximum sizes at a given time are shown in Table 5.1. We have a portfolio $\mathbf{p} = (0, 3400, 2400, 3200, 2800)$. This portfolio contains the sums of all maximum sizes for each illiquid asset, i.e., $p_i = p_i^{max}, i = 1, \ldots, 4$.

maximum Size	Bids
200	11.65
200	11.55
200	11.45
200	11.1
200	11.05
200	11
200	10.3
500	9.3
500	6.5
1000	6.46

(a) Asset A_1

maximum Size	Bids
200	19.58
600	19.5
200	19.2
200	19.15
200	19.1
200	18.6
200	18.5
200	16.85
200	16.1
200	16.05

(b) Asset A_2

maximum Size	Bids
400	29.3
200	29.16
400	29.15
400	28.9
200	28
600	27.8
200	27.15
200	27
400	26
200	22

(c) Asset A_3

maximum Size	Bids
200	43.1
400	42.65
200	41.9
400	41
200	40.86
200	40.4
200	39
400	37
400	36
200	35.1

(d) Asset A_4

Table 5.1: Bids of assets A_1-A_4

It is easy to calculate the uppermost MtM value $U(\mathbf{p})$ and the liquidation MtM value $L(\mathbf{p})$ from the tables, that is, $U(\mathbf{p}) = 3.01042 \times 10^5$ and $L(\mathbf{p}) = 2.73720 \times 10^5$. So the uppermost liquidation cost is $C(\mathbf{p}) = 0.27322 \times 10^5$. If the true portfolio value is the liquidation MtM value but if we use the uppermost MtM value instead, it will overestimate our portfolio value by as much as 10%. If we would use the artificial mid-price for calculation, then the overestimation will be definitely more than 10%.

For different cash requirements, we use the sorted marginal sensitivities (see Table 5.2) to find the liquidation sequence and then we calculate the portfolio values (see Figure 5.7). We can see that the marginal sensitivity can be as large as 44.5% for the most illiquid part of the MSDC for asset A_1, which indicates a high level of liquidity risk. We also infer that the portfolio value decreases at a faster rate as we have to liquidate positions of more illiquid assets to meet the cash requirements, which will definitely cause more losses during liquidation.

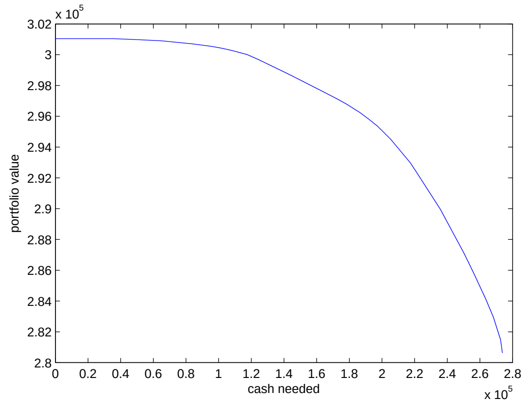


Figure 5.7: Portfolio value with different cash requirements

Asset	maximum Size	Bid	Best Bid	Marginal Sensitivity
1	200	11.65	11.65	0
2	200	19.58	19.58	0
3	400	29.3	29.3	0
4	200	43.1	43.1	0
2	600	19.5	19.58	0.004085802
3	200	29.16	29.3	0.004778157
3	400	29.15	29.3	0.005119454
1	200	11.55	11.65	0.008583691
4	400	42.65	43.1	0.010440835
3	400	28.9	29.3	0.013651877
1	200	11.45	11.65	0.017167382
2	200	19.2	19.58	0.019407559
2	200	19.15	19.58	0.021961185
2	200	19.1	19.58	0.024514811
4	200	41.9	43.1	0.027842227
3	200	28	29.3	0.044368601
1	200	11.1	11.65	0.0472103
4	400	41	43.1	0.048723898
2	200	18.6	19.58	0.050051073
3	600	27.8	29.3	0.051194539
1	200	11.05	11.65	0.051502146
4	200	40.86	43.1	0.051972158
2	200	18.5	19.58	0.055158325
1	200	11	11.65	0.055793991
4	200	40.4	43.1	0.062645012
3	200	27.15	29.3	0.07337884
3	200	27	29.3	0.078498294
4	200	39	43.1	0.09512761
3	400	26	29.3	0.112627986
1	200	10.3	11.65	0.115879828
2	200	16.85	19.58	0.139427988
4	400	37	43.1	0.141531323
4	400	36	43.1	0.164733179
2	200	16.1	19.58	0.17773238
2	200	16.05	19.58	0.180286006
4	200	35.1	43.1	0.185614849
1	500	9.3	11.65	0.201716738
3	200	22	29.3	0.249146758
1	500	6.5	11.65	0.442060086
1	1000	6.46	11.65	0.445493562

Table 5.2: Liquidation sequence

Merits and shortcomings of the calculation scheme. The calculation scheme presented above is useful, especially if we hold a portfolio of many assets with large positions. When solving the original convex optimization problem, most of the time, the equation constraints are likely to contain a series of piecewise constant functions and this might increase the difficulty for searching the global optimal solution within the domain. Instead, with the aforementioned calculation scheme we only need to calculate the marginal sensitivities with respect to different numbers of asset units traded (as well as their corresponding maximum trading sizes) as long as we have access to the bid and ask prices at a given time from real data.

This scheme shows an efficient searching direction to the optimal solution guided by the liquidation sequence. When using the four-asset example with the cash liquidity policy above, we use the calculation scheme and the the *fmincon* function in Matlab to solve the optimization problem for 2 million times and record the averaged computational time for each case. The averaged computation time of our calculation scheme is 0.0013 second, the averaged time for the *fmincon* function is 0.1733 second. We see that the calculation scheme is more than 100 times faster than the *fmincon* function in Matlab and we reach an accurate solution, compared to the local and thus possibly suboptimal solution obtained by the *fmincon* function. However, this scheme only works for an optimization problem with equality constraints, for example, under the cash liquidity policy as discussed above. When inequality constraints occur this calculation scheme may not converge.

5.3 Modeling ladder MSDCs by exponential MSDCs

From Section 5.2, we find a fast calculation scheme for portfolio valuation with ladder MSDCs. However, for general modeling purposes, we may try to use a continuous MSDC model to approximate the ladder MSDC and then apply the Lagrange multiplier method to obtain an analytical solution and to improve computational efficiency. As such, we here propose to use exponential MSDCs from Section 5.1 to model ladder MSDCs.

5.3.1 Modeling ladder MSDCs

When using exponential MSDCs to approximate ladder MSDCs, the best bids $m_i(0^+)$ can be taken as the market risk factors M_i in the exponential functions. First, we assume that the liquidity risk factors k_i are independent of the market risk factors M_i. The liquidity risk factors k_i are estimated from the ladder MSDCs by the method of least squares, as follows.

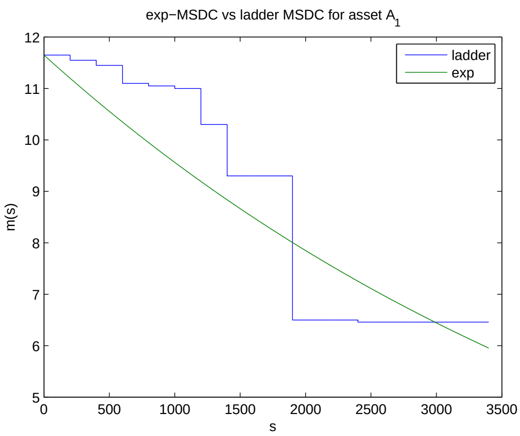
We transform the exponential function $m_i(s) = M_i e^{-k_i s}$ to $-\log(\frac{m_i(s)}{M_i}) = k_i s$. To estimate the parameter k_i, a list of n discrete pairs $(s_n, -\log(\frac{m_i(s_n)}{M_i}))$ are observed given that the market risk factor M_i is estimated beforehand. To obtain an estimate of k_i, the following sum of squares is minimized:

$$\sum_{j=1}^{n}(-\log(\frac{m_i(s_j)}{M_i}) - k_i s_j)^2$$

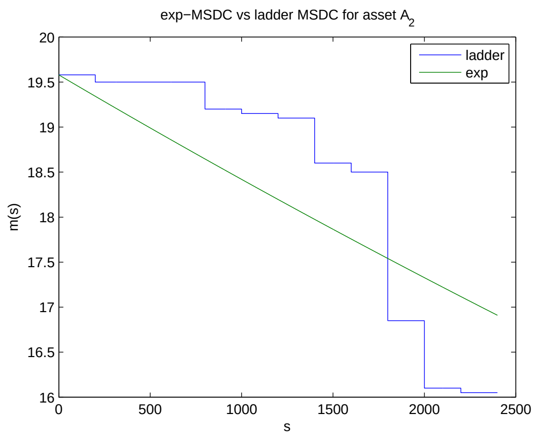
Once we solve this, then the least squares estimate of parameter k_i turns out to be

$$\widehat{k_i} = \frac{-\sum_{j=1}^{n} s_j \log(\frac{m_i(s_j)}{M_i})}{\sum_{j=1}^{n} s_j^2}$$

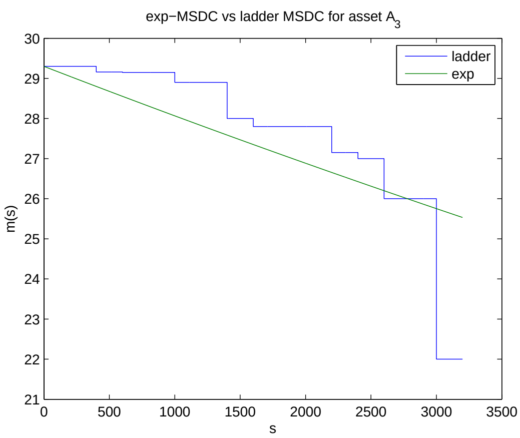
We focus on the four-asset example with ladder MSDCs from Table 5.1. In Figure 5.8 we show the ladder MSDCs and their corresponding exponential MSDCs as estimated according to the method of least squares described above. The liquidity risk factors in the exponential MSDCs turn out to be $k_1 = 1.9738 \times 10^{-4}$, $k_2 = 6.1091 \times 10^{-5}$,



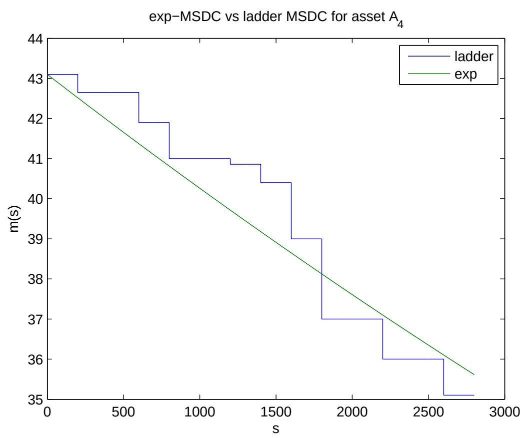
(a) Asset A_1



(b) Asset A_2



(c) Asset A_3



(d) Asset A_4

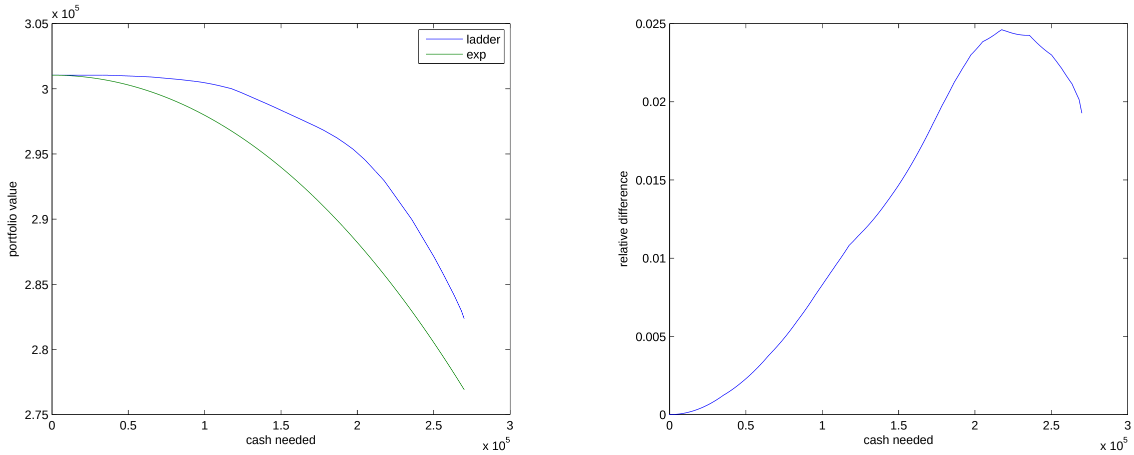
Figure 5.8: Comparison of exp MSDCs vs ladder MSDCs for the bid prices of A_1-A_4

$k_3 = 4.3015 \times 10^{-5}$ and $k_4 = 6.8139 \times 10^{-5}$. So, we conclude that asset A_1 is the most illiquid and asset A_3 is the least illiquid.

In Figure 5.9(a), we compare the approximate portfolio values from the exponential MSDCs with the accurate portfolio values from the ladder MSDCs under different cash requirements. The relative difference of portfolio values derived from these two models, i.e.,

$$\left|\frac{V_{\text{ladder}}^{\mathcal{L}(c)}(\mathbf{p}) - V_{\text{exp}}^{\mathcal{L}(c)}(\mathbf{p})}{V_{\text{ladder}}^{\mathcal{L}(c)}(\mathbf{p})}\right|$$

is shown in Figure 5.9(b). Since we find that the relative difference is at most 2.5%, we conclude that the exponential MSDCs are acceptable for modeling the original ladder MSDCs.



(a) Portfolio values from exp vs ladder MSDCs

(b) Relative difference in portfolio values

Figure 5.9: Modeling ladder MSDCs by exponential MSDCs

When using the exponential MSDCs for general modeling, we need to estimate the market risk factors M_i and the liquidity risk factors k_i from the empirical data or by some general modeling assumptions. As long as we estimate the parameters in the model, the exponential MSDCs will definitely speed up the portfolio valuation procedure in practice.

5.3.2 Modeling error

When using exponential MSDCs to model ladder MSDCs in Section 5.3.1, a natural question is to what extent the modeling is valid. This involves two issues: (1) where the modeling error comes from; and (2) when the modeling would fail.

For the first question, as the relative difference represents the error in modeling, we find that in different modeling cases the relative difference between the estimated exponential MSDC and the actual ladder MSDC has different shapes. In some cases, the relative difference is always going up, while in others it may fluctuate. We may need an indicator or a parameter to explain the modeling error. For the second question, we suspect that in some extreme cases the exponential function may fail for modeling the ladder MSDC.

To validate the model, we construct artificial extreme cases based on the concrete four-asset example which is used in Section 5.2 and 5.3.1. We modify the last part of the bid prices for all four assets to be 0.46, 0.05, 0.1 and 0.1, respectively. See Table 5.3.

maximum Size	Bids
200	11.65
200	11.55
200	11.45
200	11.1
200	11.05
200	11
200	10.3
500	9.3
500	6.5
1000	**0.46**

(a) Asset A_1

maximum Size	Bids
200	19.58
600	19.5
200	19.2
200	19.15
200	19.1
200	18.6
200	18.5
200	16.85
200	16.1
200	**0.05**

(b) Asset A_2

maximum Size	Bids
400	29.3
200	29.16
400	29.15
400	28.9
200	28
600	27.8
200	27.15
200	27
400	26
200	**0.1**

(c) Asset A_3

maximum Size	Bids
200	43.1
400	42.65
200	41.9
400	41
200	40.86
200	40.4
200	39
400	37
400	36
200	**0.1**

(d) Asset A_4

Table 5.3: Bids of assets A_1-A_4 (extreme example)

The new exponential MSDCs are shown in Figure 5.10. When we recalculate the portfolio value based on exponential and ladder MSDCs (see Figure 5.11(a)), we find that the modeling error is huge (see Figure 5.11(b)). This means that the exponential MSDC fails in this case.

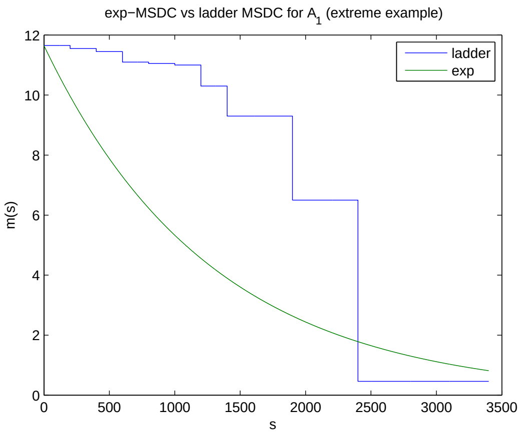
When analyzing this extreme case, we find that an important factor which determines the modeling error and the validity of exponential function relates to the huge jumps occurring in the ladder MSDCs. A huge jump increases the error in modeling. If there is no huge jump in the ladder MSDCs, the exponential MSDC can be valid for modeling. Hence, we may need a parameter to identity the jump happened in the ladder MSDC.

To this end, for asset A_i, we define a *jump indicator* $I_i(s_i)$, based on the following marginal sensitivities, as

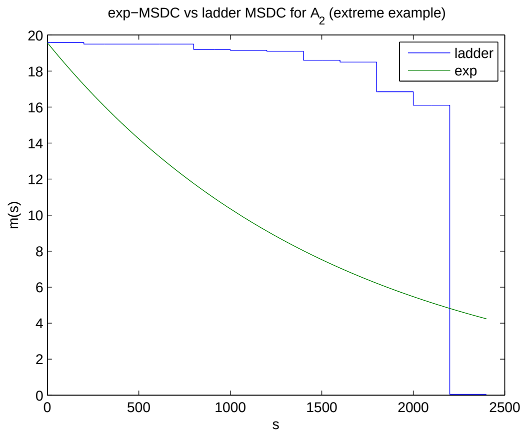
$$I_i(s_i) = MS_i(s_i^+) - MS_i(s_i^-) = \frac{m_i(s_i^-) - m_i(s_i^+)}{m_i(0^+)}$$

MS_i denotes the marginal sensitivity for asset A_i as defined in Section 5.2. The jump indicator is always non-negative as the marginal sensitivity is non-decreasing. When $I_i(s_i) = 0$, the ladder MSDC is continuous at point s_i and hence there is no jump in the MSDC at the trading volume s_i. When $I_i(s_i) > 0$, the ladder MSDC is non-continuous at s_i.

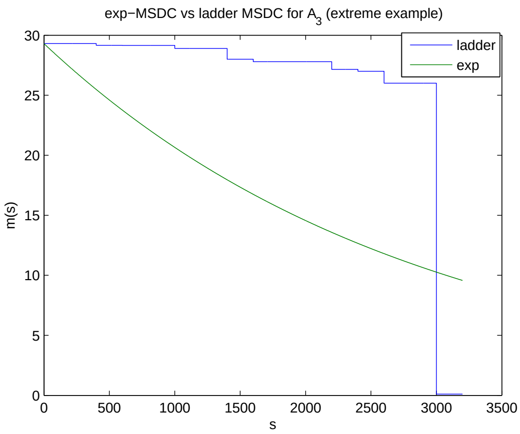
With this jump indicator, we can identify where a jump occurs and measure how large the jump is. Accordingly, we identify where a large modeling error happens. Moreover, as the jump indicator is described as a relative value, we can even compare the impact of jumps occurring in different ladder MSDCs. For the extreme example discussed



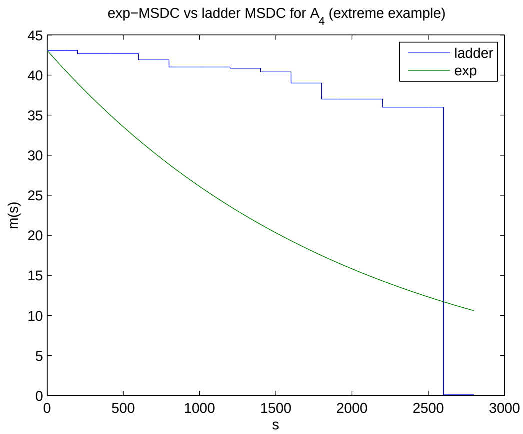
(a) Asset A_1



(b) Asset A_2



(c) Asset A_3

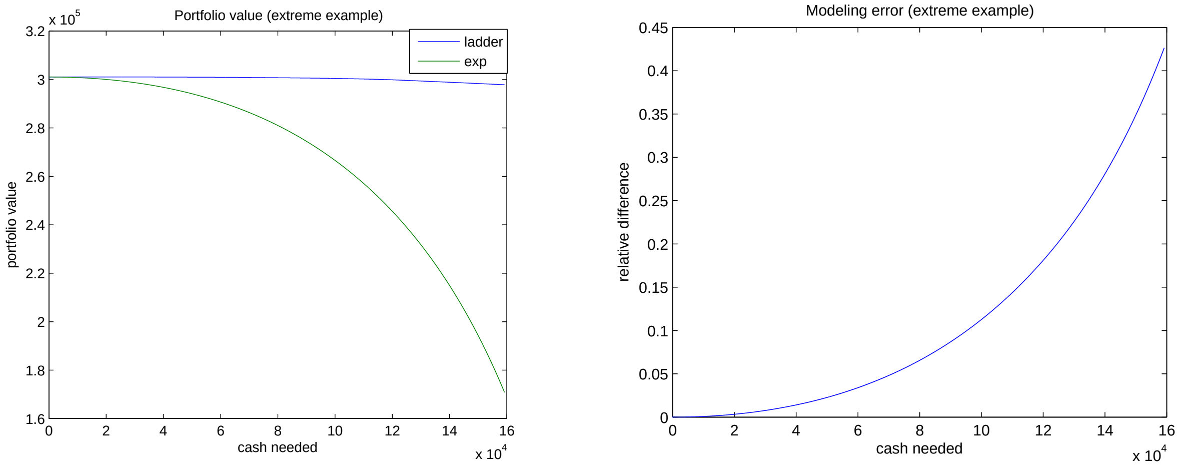


(d) Asset A_4

Figure 5.10: Comparison of exp MSDCs vs ladder MSDCs for the bid prices of A_1-A_4 (extreme example)

above, the jump indicators for asset A_1 to A_4 are shown in Figure 5.12. We can see that the largest jump occurs at the end of the ladder MSDC for asset A_3. If we use the exponential function to model the ladder MSDC for asset A_3, there will definitely be large modeling errors around the end of the ladder MSDC.

In addition, the use of the jump indicator can explain (at least partly) the shape of the relative error in the calculation of portfolio values. To do this, we first calculate marginal sensitivities for each asset. We then calculate the difference between two marginal sensitivities for each asset. That is the jump indicator at the margin of one



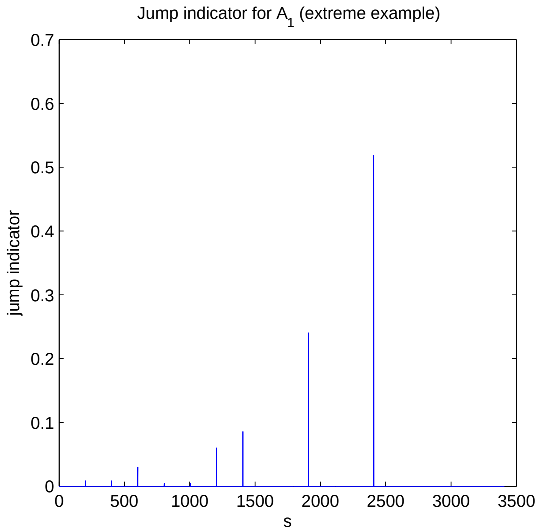
(a) Comparison of portfolio values (extreme example)

(b) Relative difference (extreme example)

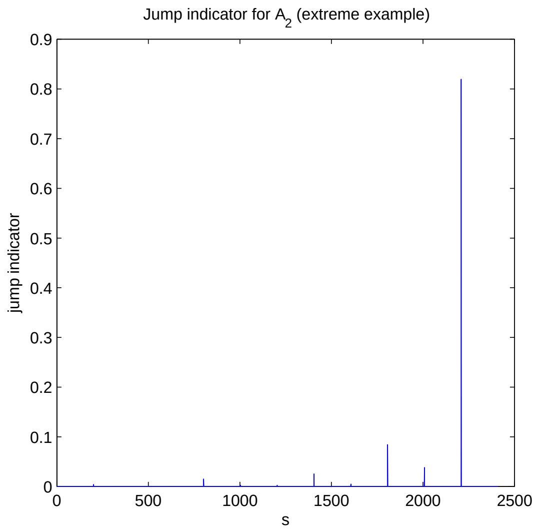
Figure 5.11: Modeling of ladder MSDCs (extreme example)

ladder of the MSDC. By sorting the non-zero marginal sensitivities in an ascending order with those non-zero jump indicators (i.e., the liquidation sequence, see Section 5.2), we find the impact of modeling error caused by the jump indicators on portfolio valuation for different liquidation requirements. For example, we can see different shapes of the modeling error in Figure 5.9(b) and 5.11(b). By sorting non-zero marginal sensitivities as well as recording corresponding non-zero jump indicators, we get a sequence of jump indicators (see Table 5.4). From Figure 5.13(a) we can see a rough trend of the jump indicators. Around the end of the graph we find the jump indicator falls to a relatively small level, which can partly explain why there is a drop of the modeling error around the end in Figure 5.9(b). An increasing trend of the jump indicator can be found in Figure 5.13(b) and this can partly account for the increasing modeling error in Figure 5.11(b) for the portfolio valuation in the extreme example.

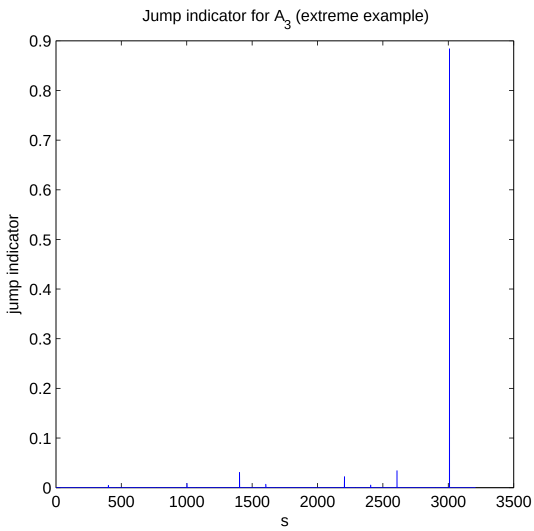
We may set a threshold for all jump indicators (say, 0.2). If one jump indicator surpasses the threshold, this indicates that a huge modeling error will happen when we use the exponential function to model the ladder MSDC. To improve modeling techniques, we may use better methods to estimate the liquidity risk factor and the market risk factor in the exponential function. Alternatively, we should find some sophisticated models to replace the exponential function in future research.



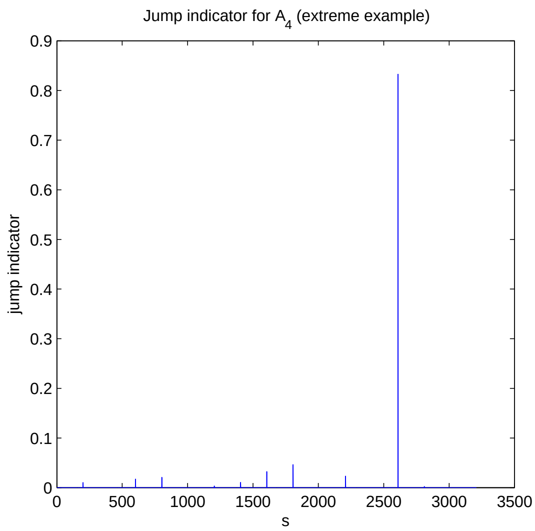
(a) Asset A_1



(b) Asset A_2



(c) Asset A_3



(d) Asset A_4

Figure 5.12: Jump indicators for the bid prices of A_1-A_4 (extreme example)

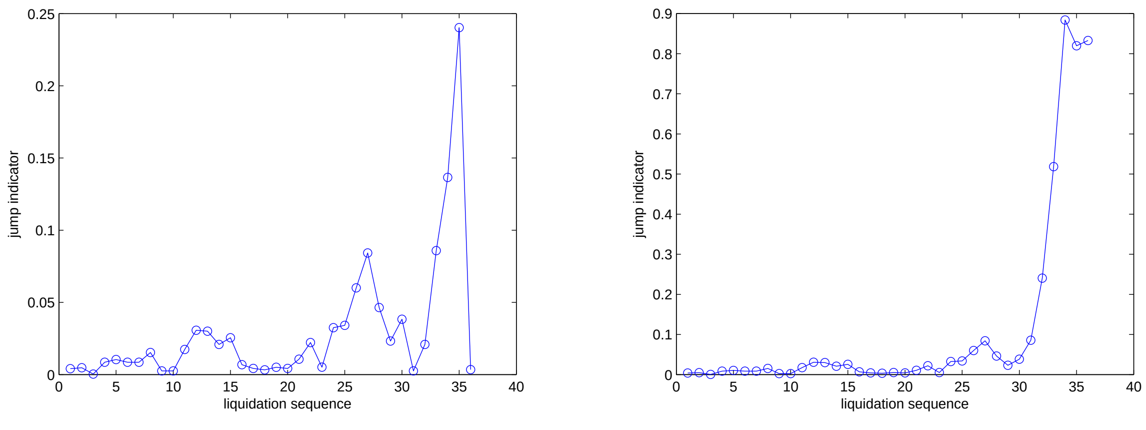
Marginal Sensitivity	Jump indicator
0.004085802	0.004085802
0.004778157	0.004778157
0.005119454	0.000341297
0.008583691	0.008583691
0.010440835	0.010440835
0.013651877	0.008532423
0.017167382	0.008583691
0.019407559	0.015321757
0.021961185	0.002553626
0.024514811	0.002553626
0.027842227	0.017401392
0.044368601	0.030716724
0.0472103	0.030042918
0.048723898	0.020881671
0.050051073	0.025536261
0.051194539	0.006825939
0.051502146	0.004291845
0.051972158	0.00324826
0.055158325	0.005107252
0.055793991	0.004291845
0.062645012	0.010672854
0.07337884	0.0221843
0.078498294	0.005119454
0.09512761	0.032482599
0.112627986	0.034129693
0.115879828	0.060085837
0.139427988	0.084269663
0.141531323	0.046403712
0.164733179	0.023201856
0.17773238	0.038304392
0.180286006	0.002553626
0.185614849	0.020881671
0.201716738	0.08583691
0.249146758	0.136518771
0.442060086	0.240343348
0.445493562	0.003433476

(a) Example in Section 5.2

Marginal Sensitivity	Jump indicator
0.004085802	0.004085802
0.004778157	0.004778157
0.005119454	0.000341297
0.008583691	0.008583691
0.010440835	0.010440835
0.013651877	0.008532423
0.017167382	0.008583691
0.019407559	0.015321757
0.021961185	0.002553626
0.024514811	0.002553626
0.027842227	0.017401392
0.044368601	0.030716724
0.0472103	0.030042918
0.048723898	0.020881671
0.050051073	0.025536261
0.051194539	0.006825939
0.051502146	0.004291845
0.051972158	0.00324826
0.055158325	0.005107252
0.055793991	0.004291845
0.062645012	0.010672854
0.07337884	0.0221843
0.078498294	0.005119454
0.09512761	0.032482599
0.112627986	0.034129693
0.115879828	0.060085837
0.139427988	0.084269663
0.141531323	0.046403712
0.164733179	0.023201856
0.17773238	0.038304392
0.201716738	0.08583691
0.442060086	0.240343348
0.960515021	0.518454936
0.996587031	0.883959044
0.997446374	0.819713994
0.997679814	0.832946636

(b) Extreme example

Table 5.4: Jump indicator sequences



(a) Example in Section 5.2 (b) Extreme example in this section

Figure 5.13: Impact of jump indicators on modeling portfolio valuation

Chapter 6

Revised Market Risk Measurement

In this chapter, we propose a quantitative definition of market liquidity risk. Conventional risk measures ignore the effect of liquidity risk, which will result in an underestimate of risk measures. To include this point, the *portfolio risk measure (PRM)* is proposed in Section 6.2. Furthermore, due to the introduction of liquidity risk, the coherent properties of a coherent risk measure will change to the convex properties for a so-called coherent portfolio risk measure (CPRM) which is induced by a coherent risk measure as we shall see in Section 6.2.2.

6.1 Quantification of market liquidity risk

For the purpose of quantifying liquidity risk, researchers mainly focus on the uncertain and variable costs but exclude the static and known costs, like commission fees and taxes. We will also follow this approach and concentrate on the uncertain and variable costs, i.e., the liquidation cost[1], when measuring liquidity risk.

From Proposition 4.4, we know that our portfolio value under any liquidity policy is not higher than the uppermost portfolio value. This implies that if we only use the uppermost MtM value as the true portfolio value we are likely to overestimate the value of our portfolio. So, can we quantitatively measure the impact of liquidity risk on our portfolio?

In Acerbi's portfolio theory, the uppermost liquidation cost $C(\mathbf{p})$ is considered (see Definition 4.8). For the purpose of taking into account liquidity risk, we focus on the liquidation cost.

[1]An act of exchange of a less liquid asset with a more liquid asset, say cash, is called liquidation.

Definitions of liquidation cost are given in the literature, using an idea of multi-period liquidation. These approaches often make use of the mid-price for calculation. For example, Buhl [7] suggests a definition of liquidation cost under a discrete liquidation strategy at times $t_1, \ldots, t_n$ as

$$L(Q) := \sum_{i=1}^{n} \int_0^{q_i} f_i(x)\mathrm{d}x \cdot e^{-r(t_i - t_1)} - Q \cdot V_1$$

with

$$\begin{aligned} q_i &= \text{trading volume at time } t_i, \\ f_i(q_i) &= \text{price-volume function at time } t_i, \\ r &= \text{risk-free interest rate}, \\ V_1 &= \text{mid-price at time } t_1, \\ Q &= \sum_{i=1}^{n} q_i. \end{aligned}$$

For a one-period trade, i.e., $n = 1$, this reads $L(Q) = \int_0^Q f_1(x)\mathrm{d}x - Q \cdot V_1$.

Another example is proposed by Loebnitz [11], where the liquidation cost[2] given a specific trading strategy k, $L^k(Q)$ is defined as

$$L^k(Q) := \sum_{i=1}^{n} q_i \cdot T_i(q_i) \cdot e^{-r(t_i - t_1)} - Q \cdot V_0$$

with

$$\begin{aligned} q_i &= \text{order size at time } t_i \text{ according to strategy } k, \\ T_i(q_i) &= \text{transaction price for the order size } q_i \text{ at time } t_i, \\ r &= \text{risk-free interest rate}, \\ V_0 &= \text{benchmark price at time zero}, \\ Q &= \sum_{i=1}^{n} q_i. \end{aligned}$$

The one-period trading version is $L^k(Q) = Q \cdot T_1(Q) - Q \cdot V_0$.

The benchmark price, V_0, in the above formulation can be chosen to be the quoted price just before the filing of the order. This can be the mid, bid or ask price. One can also

[2] In [11], the liquidation cost is called a price concession.

use an estimate as the benchmark price, especially in OTC markets, where a quoted price is usually not available.

Based on Acerbi's framework of portfolio theory, we here define the *liquidation cost*, $LC(\mathbf{p})$, as the difference between the uppermost MtM portfolio value and the true portfolio value under some liquidity policy as

$$LC(\mathbf{p}) := U(\mathbf{p}) - V^{\mathcal{L}}(\mathbf{p}).$$

Compared with the one-period trading versions of Buhl's and Loebnitz's definitions of liquidation cost, our definition has two benefits. First, $LC(\mathbf{p})$ is not only defined under the liquidating-all policy but varies with different liquidity policies. Second, it includes the use of the bid or ask price as the market price.

With this definition as a quantification of market liquidity risk, we can evaluate to which extent liquidity risk influences portfolio value under different liquidity policies. However, if we want to compare the liquidity of different portfolios, this approach can only give an absolute impact of liquidity risk and thus fails.

To this end, we define *market liquidity risk* as a relatively quantity.

Definition 6.1. The **market liquidity risk** of a portfolio $\mathbf{p}$ given a liquidity policy $\mathcal{L}$, can be mathematically defined as

$$LR(\mathbf{p}) := \left| \frac{U(\mathbf{p}) - V^{\mathcal{L}}(\mathbf{p})}{U(\mathbf{p})} \right| .$$

Market liquidity risk is measured as the relative difference between the uppermost MtM value and the present MtM value under a given liquidity policy. It measures how liquidity risk influences the MtM portfolio value. As $LR(\mathbf{p})$ is non-negative, it indicates to which extent we overvalue our portfolio if we use the uppermost MtM portfolio value as the true portfolio value. The use of the absolute value in the definition is motivated by cases where our portfolio may have a negative MtM value.

6.2 Revised risk measures

Let us extend the definition of risk measure. We do not wish to change the rules of conventional risk measures, but merely introduce a new kind of risk measure which includes the impact of the liquidity policy.

6.2.1 Portfolio risk measures

So far we have been tacitly assuming that MSDCs do not change over time. We do not normally expect this to be the case. If we assume that MSDCs exhibit a stochastic dynamics, then the value of aportfolio will also vary stochastically. Similar to the discussion on the definition of risk measure in Section 3.1, we consider a set of random varibles induced by a given liquidity policy $\mathcal{L}$ as $X^{\mathcal{L}}(\mathbf{p}) := \{X_t^{\mathcal{L}}(\mathbf{p})\}_t$.

$X_t^{\mathcal{L}}(\mathbf{p})$ can denote the portfolio value or the P&L of a portfolio under a given liquidity policy. We give the following definition of a portfolio risk measure:

Definition 6.2. Given a risk measure ρ as defined in Definition 3.1, and a liquidity policy $\mathcal{L}$, the **portfolio risk measure** $\rho^{\mathcal{L}}$ of a portfolio $\mathbf{p}$, is a function mapping portfolio space $\mathcal{P}$ to $\mathbb{R}\bigcup\{+\infty\}$, defined by

$$\rho^{\mathcal{L}}(\mathbf{p}) := \rho(X_t^{\mathcal{L}}(\mathbf{p})).$$

Note that it could happen that $X_t^{\mathcal{L}}(\mathbf{p}) = -\infty$, for instance, the portfolio value $V^{\mathcal{L}}(\mathbf{p}) = -\infty$. We will extend the portfolio risk measure to be $\rho^{\mathcal{L}}(\mathbf{p}) = +\infty$ in this case.

We can now define the *portfolio Value-at-Risk (PVaR)* under a liquidity policy $\mathcal{L}$ as

$$\mathrm{PVaR}_\alpha^{\mathcal{L}}(\mathbf{p}) := \inf\{x|\mathbb{P}[X_t^{\mathcal{L}}(\mathbf{p}) < x] \leq 1-\alpha\}$$

and define the *portfolio Expected Shortfall (PES)* as

$$\mathrm{PES}_\alpha^{\mathcal{L}}(\mathbf{p}) := \mathbb{E}[X_t^{\mathcal{L}}(\mathbf{p})|X_t^{\mathcal{L}}(\mathbf{p}) > \mathrm{PVaR}_\alpha^{\mathcal{L}}(\mathbf{p})]$$

6.2.2 Coherent portfolio risk measures

If we analyze the properties of coherent risk measure, we find some inconsistencies in an illiquid world. For example, if we double the size of our portfolio which is invested in some illiquid assets, this could increase the liquidity risk of our portfolio, and thus should increase the value of risk measure more than doubled, which contradicts the positive homogeneity of coherent risk measure.

We then propose a coherent portfolio risk measure (CPRM)[3] if a coherent risk measure (CRM) and a liquidity policy are given (cf. Section 6.2). The defining properties of a coherent risk measure change to the following properties of a CPRM.

[3]However, we may as well call this risk measure a convex portfolio risk measure as it follows the convex properties. However, this risk measure is induced by a coherent risk measure. Here we just follow Acerbi's terminology.

Theorem 6.1. *Let $\mathcal{L}$ be any liquidity policy and $\rho^{\mathcal{L}}$ be any CPRM based on the set of portfolio values at a given time, $\{V^{\mathcal{L}}\}$. Then, we have the following properties of a CPRM:*

1. *Monotonicity: for all $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}$ with $V^{\mathcal{L}}(\boldsymbol{p}) \geq V^{\mathcal{L}}(\boldsymbol{q})$, we have $\rho^{\mathcal{L}}(\boldsymbol{p}) \leq \rho^{\mathcal{L}}(\boldsymbol{q})$.*
2. *Translational subvariance: for all $\boldsymbol{p} \in \mathcal{P}$ and $e \geq 0$, we have $\rho^{\mathcal{L}}(\boldsymbol{p}+e) \leq \rho^{\mathcal{L}}(\boldsymbol{p})-e$.*
3. *Convexity: for all $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}$ and $\theta \in [0,1]$, we have $\rho^{\mathcal{L}}(\theta\boldsymbol{p} + (1-\theta)\boldsymbol{q}) \leq \theta\rho^{\mathcal{L}}(\boldsymbol{p}) + (1-\theta)\rho^{\mathcal{L}}(\boldsymbol{q})$.*

Proof. Monotonicity can be easily proved by the monotonicity of CRM.

Translational subvariance:

$$
\begin{aligned}
&\rho^{\mathcal{L}}(\mathbf{p}+e)\\
=&\rho(V^{\mathcal{L}}(\mathbf{p}+e))\\
\leq&\rho(V^{\mathcal{L}}(\mathbf{p})+e) \text{ (by Theorem 4.5 and monotonicity of CRM)}\\
=&\rho(V^{\mathcal{L}}(\mathbf{p}))-e \text{ (by translational invariance of CRM)}\\
=&\rho^{\mathcal{L}}(\mathbf{p})-e
\end{aligned}
$$

Convexity:

$$
\begin{aligned}
&\rho^{\mathcal{L}}(\theta\mathbf{p}+(1-\theta)\mathbf{q})\\
=&\rho(V^{\mathcal{L}}(\theta\mathbf{p}+(1-\theta)\mathbf{q}))\\
\leq&\rho(\theta V^{\mathcal{L}}(\mathbf{p})+(1-\theta)V^{\mathcal{L}}(\mathbf{q})) \text{ (by Theorem 4.5 and monotonicity of CRM)}\\
\leq&\rho(\theta V^{\mathcal{L}}(\mathbf{p}))+\rho((1-\theta)V^{\mathcal{L}}(\mathbf{q})) \text{ (by subadditivity of CRM)}\\
=&\theta\rho(V^{\mathcal{L}}(\mathbf{p}))+(1-\theta)\rho(V^{\mathcal{L}}(\mathbf{q})) \text{ (by positive homogeneity of CRM)}\\
=&\theta\rho^{\mathcal{L}}(\mathbf{p})+(1-\theta)\rho^{\mathcal{L}}(\mathbf{q})
\end{aligned}
$$

□

We can see that the coherent properties of the coherent risk measure change to convex properties for the coherent portfolio risk measure under any liquidity policy. These properties are independent of the chosen liquidity policy. Property 1 confirms that the monotonicity is not changed for CPRM. Property 2 indicates that adding more cash to a portfolio should increase the portfolio value and thus decrease the risk measure of the portfolio. Property 3 combines positive homogeneity and subadditivity of the coherent risk measure into a weaker property, i.e., convexity, which however also follows the principle of risk diversification.

If we also consider the liquidity policy as a function variable for the portfolio risk measure, we will have some specific properties of the risk measures relating to the use of different liquidity policies.

Proposition 6.2. *Let $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}$.*

- *Let ρ^L denote the CPRM under the liquidating-all policy $\mathcal{L}^L$. Then*

$$\rho^L(\lambda \boldsymbol{p}) \geq \lambda \rho^L(\boldsymbol{p}), \text{ if } \lambda \geq 1$$

- *Let ρ^U denote the CPRM under the liquidating-nothing policy $\mathcal{L}^U$. Then we have the following properties:*
  1. *Positive homogeneity, i.e., $\rho^U(\lambda \boldsymbol{p}) = \lambda \rho^U(\boldsymbol{p})$, if $\lambda \geq 0$.*
  2. *Subadditivity, i.e., $\rho^U(\boldsymbol{p} + \boldsymbol{q}) \leq \rho^U(\boldsymbol{p}) + \rho^U(\boldsymbol{q})$.*
  3. *Translational invariance, i.e., $\rho^U(\boldsymbol{p} + \alpha) = \rho^U(\boldsymbol{p}) - \alpha$, if $\alpha \in \mathbb{R}$.*

- *Let $\rho^{(c)}$ denote the CPRM under the cash liquidity policy $\mathcal{L}(c)$. Then we have the following properties:*
  1. *$\rho^{(c)}(\lambda \boldsymbol{p}) \leq \lambda \rho^{(c)}(\lambda \boldsymbol{p}) \leq \rho^{(\lambda c)}(\lambda \boldsymbol{p})$, if $\lambda \geq 1, p_0 \geq 0$.*
  2. *Subadditivity if at least one portfolio has positive cash, i.e.,*

$$\rho^{(c)}(\boldsymbol{p} + \boldsymbol{q}) \leq \rho^{(c)}(\boldsymbol{p}) + \rho^{(c)}(\boldsymbol{q}), \text{ if } p_0 \geq 0.$$

As for the liquidating-all policy, if a portfolio is twice the size of another portfolio, its CPRM will be more than twice the value of the other. As for the liquidating-nothing policy, all properties match those of a coherent risk measure as the constraint of the liquidating-nothing policy just replaces the mid-price in the conventional coherent risk measure by the best bid price and this does not change the rules of coherency. As for the cash liquidity policy, the subadditivity holds when at least one portfolio has positive cash.

When comparing the CPRMs under different liquidity policies for a same portfolio, we have the following result:

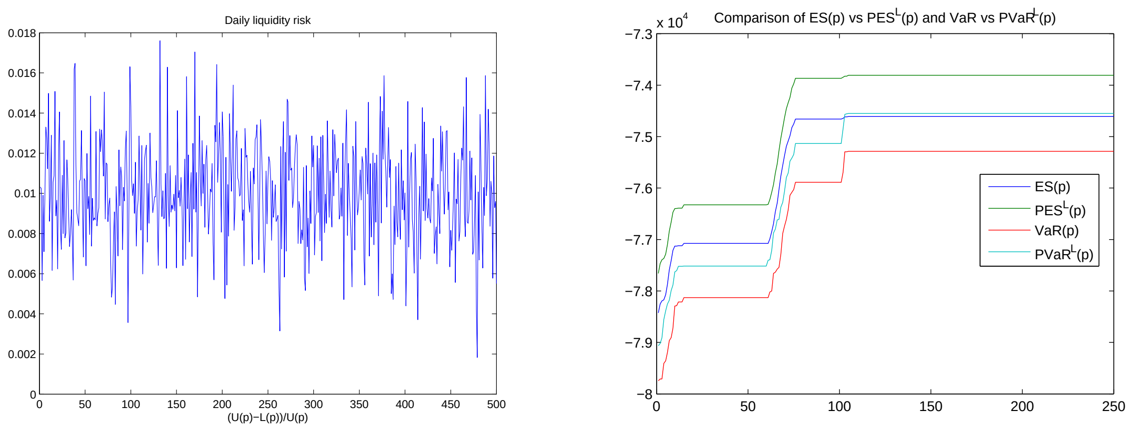
Proposition 6.3. *Let $\boldsymbol{p} \in \mathcal{P}$. If $\mathcal{L}_1$ is less restrictive than $\mathcal{L}_2$, then $\rho^{\mathcal{L}_1}(\boldsymbol{p}) \leq \rho^{\mathcal{L}_2}(\boldsymbol{p})$. Furthermore, for any liquidity policy $\mathcal{L}$, we have $\rho^U(\boldsymbol{p}) \leq \rho^{\mathcal{L}}(\boldsymbol{p})$.*

Proof. By Proposition 4.4 and the monotonic property of the conventional coherent risk measure, a proof is easily obtained. □

This proposition gives a lower bound of a CPRM. The liquidating-nothing policy $\mathcal{L}^U$ corresponds to no liquidity risk by Definition 6.1, and thus it leads to the lowest risk measure compared to all other liquidity polices.

6.3 Examples

We present some examples to obtain insight in the risk measures mentioned in Sections 6.1 and 6.2. We assume that there is one illiquid asset in our portfolio, the MSDC of which is an exponential function as discussed in Section 5.1. We furthermore assume that the best bid price follows a Geometric Brownian Motion model as $\mathrm{d}M_t = rM_t\mathrm{d}t + \sigma M_t \mathrm{d}W_t$, with $M_0 = 1, r = 0.03, \sigma = 0.2$. Using this model, we generate best bid prices for 500 trading days. We assume that the liquidity risk factor in the exponential MSDC follows a normal distribution, independent of the market risk factor. In the examples to follow, we choose two portfolios $\mathbf{p} = (0, 10000)$ and $\mathbf{q} = (0, 20000)$. We then use the daily portfolio Expected Shortfall (PES) with a confidence level of 95% defined on the set of portfolios, as an example of the CPRM. We use historical simulation by evaluating the 250 most recent daily trading data for the calculation of risk measures.



(a) Daily liquidity risk

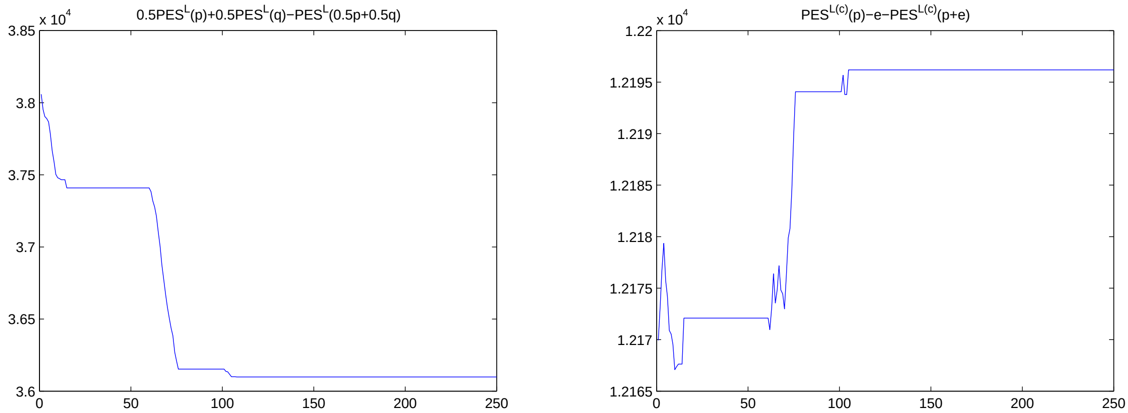
(b) Comparison of risk measures

Figure 6.1: Market liquidity risk and market risk measures

Example 6.1 (Market liquidity risk and market risk measures)**.** We set our liquidity policy here to be the liquidating-all policy and calculate market liquidity risk by $LR(\mathbf{p}) = |\frac{U(\mathbf{p})-L(\mathbf{p})}{U(\mathbf{p})}|$ (cf. Definition 6.1). From Figure 6.1(a), we see that our portfolio is overvalued by at most 1.7%, in the worst case, when liquidity risk is neglected.

Due to the inclusion of market liquidity risk, we expect market risk measures to increase compared to conventional risk measures. We compare VaR and ES with the new PVaR and PES under the liquidating-all policy. We calculate the conventional VaR and ES defined on the portfolio value only, and we use the best bid prices for calculation, rather than the mid-prices. We can see that $\mathrm{VaR}_{95\%}(\mathbf{p}) \le \mathrm{PVaR}^{L}_{95\%}(\mathbf{p})$ and $\mathrm{ES}_{95\%}(\mathbf{p}) \le \mathrm{PES}^{L}_{95\%}(\mathbf{p})$ from Figure 6.1(b). We also note that the new PVaR could even surpass the conventional ES due to the addition of liquidity risk.

Example 6.2 (General properties of CPRM)**.** First we set the liquidity policy to be the liquidating-all policy. Figure 6.2(a) shows that $\frac{1}{2}\mathrm{PES}^{L}_{95\%}(\mathbf{p})+\frac{1}{2}\mathrm{PES}^{L}_{95\%}(\mathbf{q})$ is greater than $\mathrm{PES}^{L}_{95\%}(\frac{1}{2}\mathbf{p}+\frac{1}{2}\mathbf{q})$, which confirms the convexity of the CPRM. We also calculate the CPRM with the cash liquidity policy to confirm the translational subvariance. We set our cash needed to be 40000 and the cash added to the portfolio $\mathbf{p}$ to be 12000. Figure 6.2(b) shows the translational subvariance as $\mathrm{PES}^{L}_{95\%}(\mathbf{p}+e) \le \mathrm{PES}^{L}_{95\%}(\mathbf{p}) - e$.



(a) Convexity

(b) Translational subvariance

Figure 6.2: Convexity and translational subvariance of CPRM

Chapter 7

Conclusions and Questions

7.1 Conclusions

The main aim of the thesis is to formulate a concept of liquidity risk and to incorporate market risk measurement with liquidity risk for improvement. To this end, we first review two types of liquidity risk and the relation between liquidity risk and market risk. The thesis is based on a new framework of portfolio theory introduced by Acerbi. According to this formulation, the liquidity of the assets consisting a portfolio is built into the value of that portfolio via a so-called liquidity policy. Under the new framework, the valuation of a portfolio becomes a convex optimization problem. As our own contribution, some examples of calculation schemes for the convex optimization problem are given (see Chapter 5). Equipped with the new portfolio theory, we can quantify market liquidity risk and introduce a new kind of risk measure which includes the impact of liquidity risk.

However, this will not be the end of the work. We find the new framework of portfolio theory and the quantification of liquidity risk need to be bettered by more effort.

7.2 Possible questions for future study

In what follows, we raise some further questions we are faced with in the study of liquidity risk. Here we just list a few ideas for future study.

7.2.1 Modeling of liquidity risk factor in exponential MSDC

From Chapter 5, we can see that the exponential MSDC, $m(s) = Me^{-ks}$, appears a good model in practice. The market risk factor M can be the best bid if we hold long positions. The dynamics of M can be modeled in a way similar to the dynamics for mid-price models (e.g., GBM model). The remaining work is to model the liquidity risk factor k in the exponential MSDC.

We assume that the liquidity risk factor k in the exponential MSDC is independent of the market risk factor M. We treat the liquidity risk factor as a collection of random variables over time, i.e., as a time series $\{k_t\}_t$. If there exist a linear relationship between k_t and information available prior to time t, then linear time series models such as the autoregressive (AR) model can be applied to capture the linear relationship. For example, we assume that the time series of liquidity risk factor k_t follows the AR(p) model as

$$k_t = \varphi_0 + \sum_{i=1}^{p} \varphi_i k_{t-i} + \varepsilon_t$$

where $\varphi_0, \ldots, \varphi_p$ are the parameters of the model and $\{\varepsilon_t\}_t$ is white noise.

As an example, we here use the AR(1) model to forecast liquidity risk factor in exponential MSDC at the closing time for the stock of Royal Dutch Shell. (Data come from Euronext.) We first use a dataset to approximate parameters φ_0 and φ_1 by the method of least squares to get $\widehat{\varphi}_0$ and $\widehat{\varphi}_1$. The one-step forecast $\widetilde{k}_{t+1}(1)$ given k_t is given as

$$\widetilde{k}_{t+1}(1) = \mathbb{E}[k_{t+1}|k_t] = \widehat{\varphi}_0 + \widehat{\varphi}_1 k_t$$

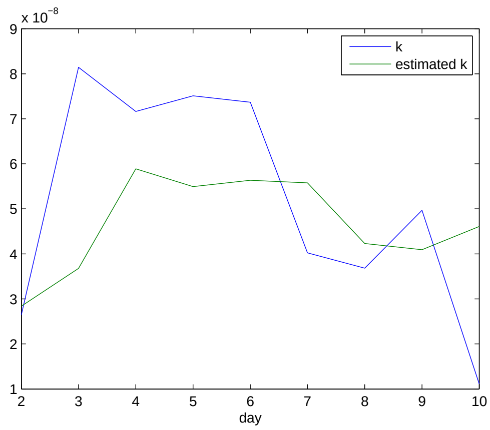


Figure 7.1: Estimated liquidity risk

In Figure 7.1 we compare the estimated liquidity risk factor with the real one for consecutive 9 trading days.

The AR(1) model may not capture all information prior to time t and the error is large. There are two ways to remedy this: using more empirical data, or using other models for approximation. We refer to [18] for details on financial time series models. Alternatively, we can use some stochastic processes to model liquidity risk. However, all these models need support from empirical data for validation.

If we assume that the liquidity risk factors are correlated with market risk factors, then a question is how to model the correlation between k and M. We provide an idea of how this might be done. We first sample from empirical data to estimate a series of liquidity risk factor independent of market risk factor. Then we use the series of estimates, $\{\widehat{k^j}\}$ and $\{\widehat{M^j}\}$, to estimate the correlation as

$$\widehat{\rho}(k, M) = \frac{\sum_{j=1}^{N}(\widehat{k^j} - \mu_k)(\widehat{M^j} - \mu_M)}{\sqrt{\sum_{j=1}^{N}(\widehat{k^j} - \mu_k)^2 \sum_{j=1}^{N}(\widehat{M^j} - \mu_M)^2}}$$

where $\mu_k = \frac{1}{N}\sum_{j=1}^{N}\widehat{k^j}$ and $\mu_M = \frac{1}{N}\sum_{j=1}^{N}\widehat{M^j}$. Then we use the correlation to modify the independent samples $\widehat{k^j}$ to be a new estimate which is correlated with M. We estimate the correlation again and modify the estimated liquidity risk factor, and so on, until certain conditions are satisfied.

7.2.2 Multi-period liquidation

One possible deficiency of Acerbi's formulation of portfolio value is that it may not be reasonable to liquidate all your positions immediately as we are possibly faced with huge losses to do this. Hence, the question arises how to define the portfolio value based on a multi-period liquidation.

We first assume that part of our portfolio $\mathbf{q}^k$ is to be liquidated at time t_k $(k = 1, \ldots, n)$. Then we add up the present uppermost MtM value of the part which is not to be liquidated and the present MtM liquidation values for $\mathbf{q}^k$. The portfolio value at time t_0 could be defined as

$$V_{t_0}(\mathbf{p}) = U_{t_0}(\mathbf{p} - \sum_{k=1}^{n}\mathbf{q}^k) + \sum_{k=1}^{n} e^{-r(t_k - t_0)} L_{t_k}(\mathbf{q}^k)$$

where r is the interest rate which is used to discount the future value to the present value.

In this definition of portfolio value, the formula no longer corresponds to an optimization problem. Furthermore, there are only two liquidity policies applied, i.e., the liquidating-all policy and the liquidating-nothing policy. Although quite unlike Acerbi's original formulation, this multi-period formula is similar to the ideas of multi-period liquidation proposed by Buhl [7] and Loebnitz [11], respectively, as discussed in Section 6.1. A natural question is: can we find an optimal liquidation scheme to minimize liquidation cost?

7.2.3 Difficulties in OTC markets

The price information in OTC markets is difficult to obtain. Hence, the assumption that all bid and ask prices can be found in the market at a given time may not hold for OTC assets. This is a serious issue, as a crucial element in Acerbi's framework is the MSDC. When prices cannot be quoted, MSDCs cannot be constructed and hence applications of the framework fail.

We hope that from the lessons of the present subprime crisis, more transparent information on the OTC assets could be revealed. However, the difficulty in discovering price information in OTC markets is yet a challenge for all researchers to study.

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